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NUMERICAL SIMULATION OF WEATHER AND CLIMATE

Technical Report No. 7

DESIGN OF THE UCLA GENERAL CIRCULATION MODEL

by

Akio Arakawa

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PREFACE

This report is an edited version of notes distributed at the Summer Workshop on the UCLA General Circulation Model in June 1971. It presents the computational schemes of the UCLA model, along with the mathematical and physical principles on which these schemes are based.

Included are the finite difference schemes for the governing fluid-dynamical equations, designed to maintain the important integral constraints and dispersion characteristics of the motion.

Also given are the principles of parameterization of cumulus convection by an ensemble of identical clouds. (A newer parameterization of cumulus convection, by an ensemble of clouds with a spectral distribution of sizes, will be published in a subsequent report.)

A model of the ground hydrology, involving the liquid, ice and snow states of water, is included.

A short summary is given of the scheme for computing solar and infrared radiation transfers through clear and cloudy air. A more detailed description of the scheme, by A. Katayama, is in press as Technical Report No. 6.

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TABLE OF CONTENTS

I. GOVERNING EQUATIONS IN σ-COORDINATE

1. σ-coordinate.

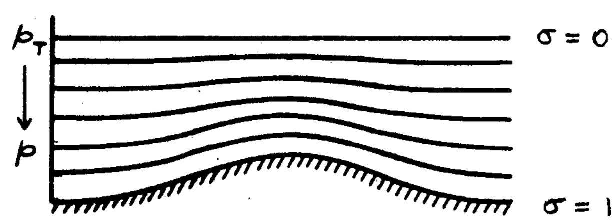
The vertical coordinate used in the model is σ, defined by

$$\sigma \equiv \frac{p - p_T}{p_S - p_T} , \tag{I.1}$$

where p_T is the pressure at the upper boundary of the model atmosphere, taken as a constant, and p_S is the pressure at the earth's surface, which is a function of the horizontal coordinates and time. It follows from (I.1) that

$$\left.\begin{array}{lll} \sigma = 0 & \text{at} & p = p_T , \\ \sigma = 1 & \text{at} & p = p_S , \end{array}\right\} \tag{I.2}$$

so that the boundaries are coordinate surfaces.



We define π by

$$\pi \equiv p_S - p_T . \tag{I.3}$$

π/g is the mass of the entire vertical column of the model atmosphere with unit horizontal cross section. From (I.1) and (I.3),

$$p = p_T + \pi\sigma . \tag{I.4}$$

From (I.4),

$$\omega \equiv \frac{dp}{dt} = \pi\dot{\sigma} + \sigma \frac{d\pi}{dt} ,$$

where $\dot{\sigma} \equiv d\sigma/dt$. Since π is a function of the horizontal coordinates and time only,

$$\omega = \pi\dot{\sigma} + \sigma(\frac{\partial}{\partial t} + \mathbb{V} \cdot \nabla)\pi , \tag{I.5}$$

where $\mathbb{V}$ is the horizontal velocity and ∇ is the horizontal gradient operator. We must remember that $\pi\dot{\sigma}$ is not identical with ω.

The earth's surface is a material surface as well as a coordinate surface. The kinematical boundary condition there is simply $\dot{\sigma} = 0$. At $\sigma = 0$ ($p = p_T$), we have $\pi\dot{\sigma} = \omega_{p=p_T}$. We may assume that $\omega_{p=p_T} = 0$ is an acceptable approximation, if p_T is chosen properly. Then

$$\dot{\sigma} = 0 \qquad \text{at} \quad \sigma = 0, 1 . \tag{I.6}$$

2. <u>The hydrostatic equation</u>.

When pressure is the vertical coordinate, we have

$$\frac{\partial \Phi}{\partial p} = -\alpha , \tag{I.7}$$

where $\Phi \equiv gz$ and α is the specific volume. From (I.4),

$$\delta p = \pi \, \delta\sigma \ , \tag{1.8}$$

where δ denotes the differential under constant horizontal coordinates and time (and, therefore, under constant π). (1.7) and (1.8) give

$$\delta\Phi = -\pi \alpha \, \delta\sigma \ . \tag{1.9}$$

The following alternative forms of the hydrostatic equation can be derived from (1.9) and will be useful:

$$\delta(\Phi\sigma) = (\Phi - \sigma \pi \alpha)\delta\sigma \ , \tag{1.10}$$

$$\delta\Phi = -c_p \theta \, \delta(p^{\varkappa}) \ , \tag{1.11}$$

$$\delta(c_p T + \Phi) = p^{\varkappa} c_p \, \delta\theta \ , \tag{1.12}$$

where $\varkappa \equiv R/c_p$ and $\theta \equiv T/p^{\varkappa}$.

3. <u>The equation of continuity.</u>

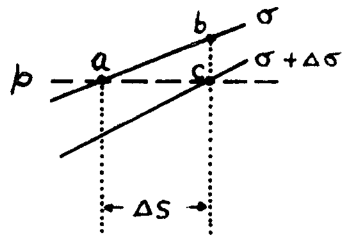
In the pressure coordinate system, the equation of continuity takes the form

$$\nabla_p \cdot \mathbb{V} + \frac{\partial \omega}{\partial p} = 0 \ . \tag{1.13}$$

We have the relation

$$\nabla_p = \nabla_\sigma + (\nabla_p \sigma)\frac{\partial}{\partial \sigma} \ . \tag{1.14}$$

This relation is illustrated in the accompanying figure.



Let A be an arbitrary quantity. Then

$$\frac{A_c - A_a}{\Delta s} = \frac{A_b - A_a}{\Delta s} + \frac{A_c - A_b}{\Delta \sigma}\,\frac{\Delta \sigma}{\Delta s}\ .$$

The limit as $\Delta s \to 0$ gives

$$\nabla_p A = \nabla_\sigma A + \frac{\partial A}{\partial \sigma}\,\nabla_p \sigma\ .$$

From (1.1) and (1.3), we obtain

$$\nabla_p \sigma = \nabla_p \left(\frac{p - p_T}{\pi}\right) = -\frac{p - p_T}{\pi^2}\,\nabla \pi = -\frac{\sigma}{\pi}\,\nabla \pi\ .$$

Then (I.14) gives

$$\nabla_p = \nabla_\sigma - \frac{\sigma}{\pi}\,\nabla \pi\,\frac{\partial}{\partial \sigma}\ . \tag{I.15}$$

Using (I.15) for $\nabla_p \cdot \mathbb{V}$ and (I.5) and (I.8) for $\partial \omega / \partial p$, results in

$$\left[\nabla_\sigma \cdot \mathbb{V} - \frac{\sigma}{\pi}\,\nabla \pi \cdot \frac{\partial \mathbb{V}}{\partial \sigma}\right] + \frac{\partial}{\pi \partial \sigma}\left[\pi \dot{\sigma} + \sigma\left(\frac{\partial}{\partial t} + \mathbb{V} \cdot \nabla\right)\pi\right] = 0\ ,$$

and finally

$$\frac{\partial \pi}{\partial t} + \nabla_\sigma \cdot (\pi \mathbb{V}) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma}) = 0\ . \tag{I.16}$$

This form of the continuity equation, with the σ-coordinate, is similar to the continuity equation with the z-coordinate,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbb{V}) + \frac{\partial}{\partial z}(\rho \omega) = 0 \ ,$$

where ρ is density. Let ΔS be a horizontal area element. Then $\rho\ \Delta z\ \Delta S$ is the mass of the volume element $\Delta z\, \Delta S$. Similarly, $(\pi/g)\Delta\sigma\, \Delta S$ is the mass of the "volume" element $\Delta\sigma\, \Delta S$, in σ-space. However, π is a function of the horizontal coordinates and time only, whereas ρ may also be a function of the vertical coordinate.

The equation of continuity (I.16) is used for two purposes: to find $\partial\pi/\partial t$ (which is $\partial p_s/\partial t$) and to find $\dot{\sigma}$.

Integrating (I.16) with respect to σ, from 0 to 1, and using the boundary condition (I.6), we obtain

$$\frac{\partial \pi}{\partial t} = - \int_0^1 \nabla_\sigma \cdot (\pi \mathbb{V})\, d\sigma \tag{I.17}$$

$$= - \nabla \cdot \int_0^1 \pi \mathbb{V} d\sigma \ . \tag{I.17'}$$

This is equivalent to the so-called "surface pressure tendency equation".

Substituting $\partial\pi/\partial t$, obtained from (I.17), into (I.16), we get $\partial(\pi\dot{\sigma})/\partial\sigma$. Integration of this derivative with respect to σ, from 0 (or 1) to a given σ, gives $\pi\dot{\sigma}$ at that σ-level.

4. <u>The individual time derivative and its flux form</u>.

With the σ-coordinate, the individual time derivative d/dt is expressed as

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_\sigma + \mathbb{V}\cdot\nabla_\sigma + \dot{\sigma}\frac{\partial}{\partial \sigma} \quad . \tag{I.18}$$

Let A be an arbitrary scalar. By using the continuity eq. (I.16), we obtain the flux forms

$$\pi\frac{dA}{dt} = \left(\frac{\partial}{\partial t}\right)_\sigma (\pi A) + \nabla_\sigma\cdot(\pi\,\mathbb{V}A) + \frac{\partial}{\partial\sigma}(\pi\dot{\sigma}A) \quad , \tag{I.19}$$

and

$$\pi A\frac{dA}{dt} = \left(\frac{\partial}{\partial t}\right)_\sigma \left(\pi\tfrac{1}{2}A^2\right) + \nabla_\sigma\cdot\left(\pi\,\mathbb{V}\tfrac{1}{2}A^2\right) + \frac{\partial}{\partial\sigma}\left(\pi\dot{\sigma}\tfrac{1}{2}A^2\right) \quad . \tag{I.19$'$}$$

In (I.19)', A may be a vector.

5. The equation of motion.

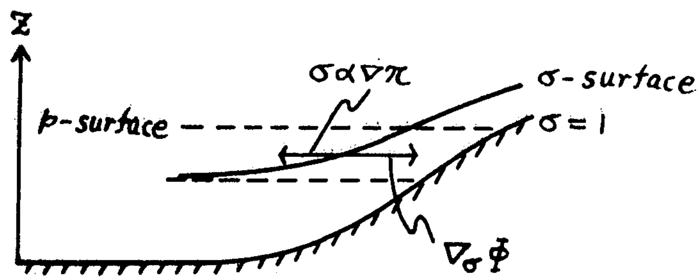
The pressure gradient force is given by $-\nabla_p\Phi$. Applying (I.15) to Φ, we obtain

$$\nabla_p\Phi = \nabla_\sigma\Phi - \frac{\sigma}{\pi}\nabla\pi\frac{\partial\Phi}{\partial\sigma} \quad , \tag{I.20}$$

and substituting from (I.9) into (I.20),

$$\nabla_p\Phi = \nabla_\sigma\Phi + \sigma\alpha\nabla\pi \quad . \tag{I.21}$$

In the σ-coordinate system, the pressure gradient force consists of two terms, as shown by equation (I.21). Where the slope of the earth's surface is steep, the individual terms are large but are approximately in the opposite directions. In the particular case where $\nabla_p\Phi = 0$, complete compensation occurs, as shown in the figure below.



The horizontal component of the equation of motion is

$$\frac{d\mathbb{V}}{dt} + f\mathbb{k} \times \mathbb{V} + \nabla_\sigma \Phi + \sigma\alpha\nabla\pi = \mathbb{F} \quad , \tag{I.22}$$

where $\mathbb{F}$ is the horizontal frictional force and $d\mathbb{V}/dt$ is the horizontal acceleration.

Note that

$$\pi(\nabla_\sigma \Phi + \sigma\alpha\nabla\pi) = \nabla_\sigma(\pi\Phi) - (\Phi - \sigma\alpha\pi)\nabla\pi \,.\,.$$

Using (I.10), $\Phi - \sigma\alpha\pi = \partial(\Phi\sigma)/\partial\sigma$. This gives us another form of the equation of motion,

$$\pi\frac{d\mathbb{V}}{dt} + f\mathbb{k} \times \pi\mathbb{V} + \nabla_\sigma(\pi\Phi) - \frac{\partial}{\partial\sigma}(\Phi\sigma)\nabla\pi = \pi\mathbb{F} \,, \tag{I.22'}$$

6. <u>The equation of state.</u>

$$\alpha = \frac{RT}{p} \,. \tag{I.23}$$

7. <u>The first law of thermodynamics.</u>

The specific entropy is $c_p \ln\theta$ + const., where $\theta \equiv T/p^{\kappa}$. The first law of thermodynamics is

$$\frac{d}{dt} c_p \ln \theta = \frac{Q}{T} \ , \tag{I.24}$$

or

$$\frac{d\theta}{dt} = \frac{1}{c_p} \frac{\theta}{T} Q \ , \tag{I.24)'$$

where Q is the heating rate per unit mass.

The flux form which corresponds to (I.24)' is

$$\frac{\partial}{\partial t}(\pi \theta) + \nabla_\sigma \cdot (\pi \mathbb{V} \theta) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} \theta) = \frac{\pi}{c_p} \frac{\theta}{T} Q \ . \qquad (I.24)''$$

The first law of thermodynamics can also be written as

$$\frac{d}{dt} c_p T = \omega \alpha + Q \ , \qquad (I.25)$$

where

$$\omega = \pi \dot{\sigma} + \sigma\left(\frac{\partial}{\partial t} + \mathbb{V} \cdot \nabla\right) \pi \ ,$$

as was given by (I.5).

The corresponding flux form is

$$\frac{\partial}{\partial t}(\pi c_p T) + \nabla_\sigma \cdot (\pi \mathbb{V} c_p T) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} c_p T) = \pi \omega \alpha + \pi Q \ . \qquad (I.25)'$$

Using (I.5), (I.25)' may be rewritten as

$$\frac{\partial}{\partial t}(\pi c_p T) + \nabla_\sigma \cdot (\pi \mathbb{V} c_p T) + p^{\kappa} \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} c_p \theta) = \pi \sigma \alpha\left(\frac{\partial}{\partial t} + \mathbb{V} \cdot \nabla\right)\pi + \pi Q \ . \qquad (I.26)$$

8. The water vapor equation.

Let q be the water vapor mixing ratio. The continuity equation for water vapor is

$$\frac{dq}{dt} = -C + E \quad , \tag{I.27}$$

where C is the rate of condensation and E is the rate of evaporation per unit mass of dry air. The corresponding flux form is

$$\frac{\partial}{\partial t}(\pi q) + \nabla_\sigma \cdot (\pi \mathbb{V} q) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} q) = \pi(-C + E) \ . \tag{I.28}$$

II. INTEGRAL PROPERTIES

The following integral properties of the governing equations, or of selected terms in these equations, are useful in designing the vertical finite difference scheme.

1. Mass conservation.

(I.17)' gives

$$\frac{\partial p_s}{\partial t} = -\nabla \cdot \int_0^1 \pi \mathbb{V} d\sigma \ . \tag{II.1}$$

The area integral of (II.1) over the entire globe vanishes. This shows that the total mass of the model atmosphere is conserved.

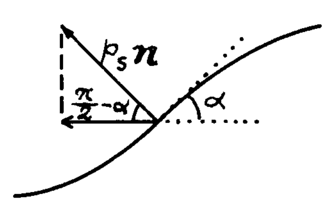
2. Vertically integrated horizontal pressure gradient force.

With the z-coordinate, the horizontal pressure force per unit mass is $-\frac{1}{\rho}\nabla_z p$, and per unit volume it is $-\nabla_z p$. Integrating vertically,

$$-\int_{z_s}^{\infty} \nabla_z p\,dz = -\left[\nabla \int_{z_s}^{\infty} p\,dz + p_s \nabla z_s\right], \tag{II.2}$$

where z_s is the height of the earth's surface.

A line integral of the tangential component of the first term on the right in (II.2), taken along an arbitrary closed curve on the sphere, always vanishes; and only the second term can contribute to the line integral. Therefore (except for the possible effect of a surface stress), only when there is a non-horizontal boundary surface can there be a "spin-up" or "spin-down" of the atmosphere along the arbitrary curve.



The accompnaying figure is a vertical cross section of the earth's topography in the plane of its slope. $p_s n$ is the pressure force normal to the surface and the horizontal component of this force is $p_s \cos(\frac{\pi}{2} - \alpha) = p_s \sin\alpha$. This is the horizontal force per unit area of the surface. Per unit horizontal area, we have $p_s \sin\alpha/\cos\alpha = p_s \tan\alpha = p_s |\nabla z_s|$.

With the p-coordinate, the horizontal pressure gradient force per unit mass is $-\nabla_p \Phi$. Its vertical integration, with respect to mass, is

$$-\frac{1}{g}\int_0^{p_s} \nabla_p \Phi \, dp = -\frac{1}{g}\left[\nabla \int_0^{p_s} \Phi dp - \Phi_s \nabla p_s\right] \qquad (II.3)$$

$$= -\frac{1}{g}\left[\nabla \int_0^{p_s} (\Phi - \Phi_s) dp + p_s \nabla \Phi_s\right].$$

With the σ-coordinate, if we start with the form of the horizontal pressure gradient force given in (I.22)', we immediately obtain the equivalent relation

$$-\frac{1}{g}\int_0^1 \left[\nabla_\sigma(\pi \Phi) - \frac{\partial}{\partial\sigma}(\Phi\sigma)\nabla\pi\right] d\sigma = -\frac{1}{g}\left[\nabla\int_0^1 \pi(\Phi - \Phi_s)d\sigma + \pi\nabla\Phi_s\right]. \quad (II.4)$$

In order to derive (II.4) from the form given by (I.22), we must use the relation

$$\Phi_s = \int_0^1 (\Phi - \sigma\pi\alpha) d\sigma , \qquad (II.5)$$

which is obtained from (I.10). (II.5) can be rewritten as

$$\int_0^1 (\Phi - \Phi_s) d\sigma = \int_0^1 \sigma\pi\alpha d\sigma . \qquad (II.5)'$$

When $p_T = 0$, (II.5)' is equivalent to the familiar relationship between the vertically integrated geopotential and internal energies.

3. The kinetic energy equation.

From the equation of motion (I.22), we obtain

$$\pi \frac{d}{dt}\frac{1}{2}\mathbb{V}^2 = -\pi\mathbb{V}\cdot\left[\nabla_\sigma \Phi + \sigma\alpha\nabla\pi\right] + \pi\,\mathbb{V}\cdot\mathbb{F}\ , \tag{II.6}$$

The left hand side of (II.6) can be written in the flux form (I.19)'.

The rate of kinetic energy generation by the pressure gradient force is

$$\begin{aligned}
-\pi\mathbb{V}\cdot\left[\nabla_\sigma\Phi + \sigma\alpha\nabla\pi\right] &= -\nabla_\sigma\cdot(\pi\mathbb{V}\Phi) + \Phi\nabla_\sigma\cdot(\pi\mathbb{V}) - \sigma\pi\alpha\mathbb{V}\cdot\nabla\pi \\
&= -\nabla_\sigma\cdot(\pi\mathbb{V}\Phi) - \Phi\left[\frac{\partial\pi}{\partial t} + \frac{\partial}{\partial\sigma}(\pi\dot\sigma)\right] - \sigma\pi\alpha\,\mathbb{V}\cdot\nabla\pi \\
&= -\nabla_\sigma\cdot(\pi\mathbb{V}\Phi) - \frac{\partial}{\partial\sigma}(\pi\dot\sigma\Phi) - \Phi\frac{\partial\pi}{\partial t} + \pi\dot\sigma\frac{\partial\Phi}{\partial\sigma} - \sigma\pi\alpha\,\mathbb{V}\cdot\nabla\pi \\
&= -\nabla_\sigma\cdot(\pi\mathbb{V}\Phi) - \frac{\partial}{\partial\sigma}(\pi\dot\sigma\Phi) - (\Phi - \sigma\pi\alpha)\frac{\partial\pi}{\partial t} \\
&\qquad - \pi\left[\sigma\left(\frac{\partial\pi}{\partial t} + \mathbb{V}\cdot\nabla\pi\right) + \pi\dot\sigma\right]\alpha \\
&= -\nabla_\sigma\cdot(\pi\mathbb{V}\Phi) - \frac{\partial}{\partial\sigma}\left(\pi\dot\sigma\Phi + \Phi\sigma\frac{\partial\pi}{\partial t}\right) - \pi\omega\alpha\ .
\end{aligned} \tag{II.7}$$

Here (I.16), (I.9), (I.10) and (I.5) were used. The vertical integration of the last form of (II.7) is

$$-\frac{1}{g}\int_0^1 \pi\mathbb{V}\cdot\left[\nabla_\sigma\Phi + \sigma\alpha\nabla\pi\right]d\sigma = -\frac{1}{g}\nabla\cdot\int_0^1 \pi\mathbb{V}\Phi\,d\sigma - \frac{1}{g}\Phi_s\frac{\partial\pi}{\partial t} - \frac{1}{g}\int_0^1 \pi\omega\alpha\,d\sigma\ . \tag{II.8}$$

4. Conservation of total energy.

From (II.6), (II.7) and (I.25)', we obtain

$$\left(\frac{\partial}{\partial t}\right)_\sigma (\pi \tfrac{1}{2} \mathbb{V}^2) + \nabla_\sigma \cdot (\pi \mathbb{V} \tfrac{1}{2} \mathbb{V}^2) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} \tfrac{1}{2} \mathbb{V}^2) = -\pi \mathbb{V} \cdot \left[\nabla_\sigma \Phi + \sigma \alpha \nabla \pi\right] + \pi \mathbb{V} \cdot \mathbb{F}, \quad \text{(II.9)}$$

$$\nabla_\sigma \cdot (\pi \mathbb{V} \Phi) + \frac{\partial}{\partial \sigma}\left[\left(\sigma \frac{\partial \pi}{\partial t} + \pi \dot{\sigma}\right) \Phi\right] = \pi \mathbb{V} \cdot \left[\nabla_\sigma \Phi + \sigma \alpha \nabla \pi\right] - \pi \omega \alpha , \quad \text{(II.10)}$$

and

$$\frac{\partial}{\partial t}(\pi c_p T) + \nabla_\sigma \cdot (\pi \mathbb{V} c_p T) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} c_p T) = \pi Q + \pi \omega \alpha . \quad \text{(II.11)}$$

Taking the sum of (II.9), (II.10) and (II.11), and integrating with respect to σ, from 0 to 1, gives

$$\frac{\partial}{\partial t}\left[\pi \Phi_s + \int_0^1 \pi(\tfrac{1}{2} \mathbb{V}^2 + c_p T) d\sigma\right] + \nabla \cdot \int_0^1 \pi \mathbb{V}(\tfrac{1}{2} \mathbb{V}^2 + c_p T + \Phi) d\sigma$$

$$= \int_0^1 \pi(\mathbb{V} \cdot \mathbb{F} + Q) d\sigma . \quad \text{(II.12)}$$

The area integral of (II.12) over the entire globe makes the contribution of the divergence term vanish, and we have the conservation of total energy when $\mathbb{F} = 0$ and $Q = 0$.

5. <u>Integral constraints on θ and θ^2.</u>

Under an adiabatic process, we have $d\theta/dt = 0$. The corresponding flux form, given by (I.24)", is

$$\frac{\partial}{\partial t}(\pi \theta) + \nabla_\sigma \cdot (\pi \mathbb{V} \theta) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma} \theta) = 0 . \quad \text{(II.13)}$$

Integration of (II.13) with respect to σ, from 0 to 1, gives

$$\frac{\partial}{\partial t} \int_0^1 \pi \theta d\sigma + \nabla \cdot \int_0^1 \pi \mathbb{V} \theta d\sigma = 0 . \quad \text{(II.14)}$$

Because the second term in (II.14) vanishes when the area integral is taken over the entire globe, we see that the global integral of θ, with respect to mass, is conserved.

Because $(d/dt)\theta^2 = 0$ under an adiabatic process, we can similarly derive

$$\frac{\partial}{\partial t}\int_0^1 \pi\,\theta^2\,d\sigma + \nabla\cdot\int_0^1 \pi\,\mathbb{V}\,\theta^2\,d\sigma = 0 \quad . \tag{II.15}$$

Again the second term vanishes when the area integral is taken over the entire globe so that the global integral of θ^2, with respect to mass, is conserved.

Under an adiabatic process, the frequency distribution of potential temperature does not change with time. The integral constraints are not sufficient to maintain the frequency distribution of θ, but they effectively maintain its variance as well as its mean.

6. Introduction to vertical differencing.

The integral properties discussed above will be used for the design of the vertical finite difference scheme that is presented in the next chapter.

The solutions obtained with any convergent scheme will satisfy these integral properties in the limit as the vertical grid size approaches zero. But the solutions obtained with these various schemes approach the true solution through different paths in a function space. Our aim is to find that scheme whose solution approaches the true solution through that path along which the finite difference analogs of the integral properties are maintained regardless of the grid size.

The principle that will be used here is similar to what is usually done when approximations are introduced into the governing equations. For example, when the hydrostatic equation is used as an approximation, we also use an approximate form of the horizontal component of the equation of motion, so that certain integral properties, such as energy conservation, are maintained. In doing that, however, the definition of energy is changed from its original definition. This modified energy approaches the true energy as the accuracy of the hydrostatic approximation increases.

Many schemes which do not maintain the integral constraints on θ and θ^2 were designed and tested. But none of these gave better results with long term integrations than the scheme, described in the next chapter, which does maintain these integral constraints.

III. VERTICAL DIFFERENCING, PART A

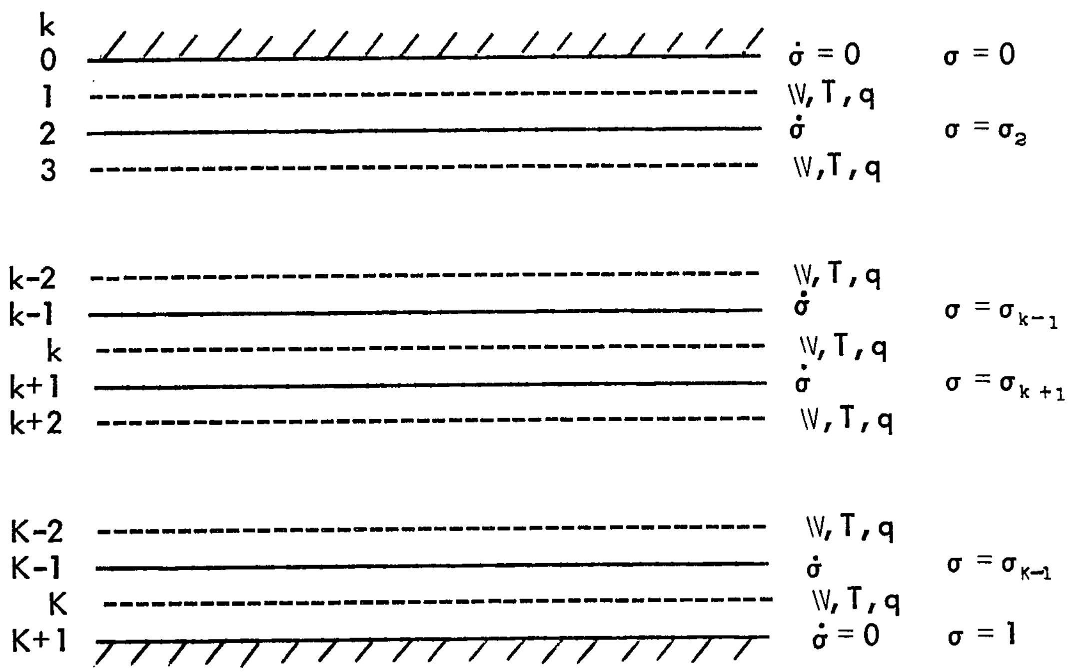
1. The vertical index.

The index k is used to identify a level.

At levels with odd k, $\mathbf{V}$, T and q are carried.

At levels with even k, $\dot{\sigma}$ is carried.

The upper boundary is k = 0 and the lower boundary is K+1.



We define
$$\Delta\sigma_k \equiv \sigma_{k+1} - \sigma_{k-1} \ . \tag{III.1}$$

Then $\sum\limits_{k=1}^{K}{}' \Delta\sigma_k = 1$, where Σ' is summation over odd k. We also define
$$\sigma_k = \tfrac{1}{2}(\sigma_{k+1} + \sigma_{k-1}) \ . \tag{III.1)'$$

The current UCLA 3-level model (December, 1971) uses $p_T = 100$ mb, $K = 5$, $\Delta\sigma_1 = 3/9$, $\Delta\sigma_3 = 4/9$ and $\Delta\sigma_5 = 2/9$.

2. The equation of continuity.

We use the form

$$\frac{\partial \pi}{\partial t} + \nabla \cdot (\pi \mathbb{V}_k) + \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1} - \pi\dot{\sigma}_{k-1}) = 0 \quad . \qquad \text{(III.2)}$$

$\sum\limits_{k=1}^{K}{}' \text{(III.2)}\, \Delta\sigma_k$ gives

$$\frac{\partial \pi}{\partial t} = - \sum_{k=1}^{K}{}' \nabla \cdot (\pi \mathbb{V}_k)\, \Delta\sigma_k \qquad \text{(III.3)}$$

which is an analog of (I.17).

$\pi\dot{\sigma}_{k+1}$ is given by

$$\pi\dot{\sigma}_{k+1} = - \sum_{k=1}^{k}{}' \left[\frac{\partial \pi}{\partial t} + \nabla \cdot (\pi \mathbb{V}_k)\right] \Delta\sigma_k \quad . \qquad \text{(III.4)}$$

3. Flux forms.

For a variable A, carried at odd levels, flux forms analogous to (I.19) can be written as

$$\frac{\partial}{\partial t}(\pi A_k) + \nabla \cdot (\pi \mathbb{V}_k A_k) + \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1} A_{k+1} - \pi\dot{\sigma}_{k-1} A_{k-1}) \quad . \qquad \text{(III.5)}$$

Because A is a variable carried at odd levels, we must somehow determine A at even levels (such as A_{k+1}, A_{k-1}).

From (III.5) and (III.2) we get

$$\pi \left[\frac{\partial A_k}{\partial t} + \mathbb{V}_k \cdot \nabla A_k + \frac{1}{\Delta\sigma_k}\Big(\dot{\sigma}_{k+1}(A_{k+1} - A_k) + \dot{\sigma}_{k-1}(A_k - A_{k-1})\Big)\right] . \qquad \text{(III.6)}$$

The expression in the bracket gives the form for dA/dt which is consistent with the flux form (III.5).

If we require that $\pi A\, dA/dt$ also be written in a flux form corresponding to (I.19)', then A at even levels must be chosen properly. (III.6) multiplied by A_k gives

$$\pi\left[\frac{\partial}{\partial t}\tfrac{1}{2}A_k^2 + \mathbb{V}_k\cdot\nabla\tfrac{1}{2}A_k^2 + \frac{1}{\Delta\sigma_k}\left(\dot{\sigma}_{k+1}(A_{k+1}A_k - A_k^2) + \dot{\sigma}_{k-1}(A_k^2 - A_kA_{k-1})\right)\right]. \qquad \text{(III.7)}$$

Using the equation of continuity, (III.7) can be rewritten as

$$\frac{\partial}{\partial t}(\pi\tfrac{1}{2}A_k^2) + \nabla\cdot(\pi\,\mathbb{V}_k\tfrac{1}{2}A_k^2) + \frac{1}{\Delta\sigma_k}\Big[\pi\dot{\sigma}_{k+1}(A_{k+1}A_k - \tfrac{1}{2}A_k^2) - \pi\dot{\sigma}_{k-1}(A_{k-1}A_k - \tfrac{1}{2}A_k^2)\Big]\,. \qquad \text{(III.8)}$$

In order that (III.8) be in flux form, $A_{k+1}A_k - \frac{1}{2}A_k^2$ must be $\frac{1}{2}A^2$ at level k+1 and $A_{k-1}A_k - \frac{1}{2}A_k^2$ must be $\frac{1}{2}A^2$ at level k-1. Therefore, $A_{k-1}A_k - \frac{1}{2}A_k^2$, with k replaced by k+2, must be equal to $A_{k+1}A_k - \frac{1}{2}A_k^2$. Then

$$A_{k+1}A_{k+2} - \tfrac{1}{2}A_{k+2}^2 = A_{k+1}A_k - \tfrac{1}{2}A_k^2\,,$$

or

$$A_{k+1}(A_{k+2} - A_k) = \tfrac{1}{2}(A_{k+2}^2 - A_k^2)\,.$$

Because $A_{k+2} - A_k$ is generally not zero, we must have

$$A_{k+1} = \tfrac{1}{2}(A_k + A_{k+2})\,. \qquad \text{(III.9)}$$

4. <u>The acceleration term</u>.

Following (III.6), we write the acceleration term as

$$\left(\frac{d\mathbb{V}}{dt}\right)_k \Rightarrow \frac{\partial\mathbb{V}_k}{\partial t} + (\mathbb{V}_k\cdot\nabla)\,\mathbb{V}_k + \frac{1}{\Delta\sigma_k}\left[\dot{\sigma}_{k+1}(\mathbb{V}_{k+1} - \mathbb{V}_k) + \dot{\sigma}_{k-1}(\mathbb{V}_k - \mathbb{V}_{k-1})\right]. \qquad \text{(III.10)}$$

To have a flux form for $\mathbb{V}_k\cdot(d\mathbb{V}/dt)_k$, we choose

$$\mathbb{V}_{k+1} = \tfrac{1}{2}(\mathbb{V}_k + \mathbb{V}_{k+2})\,. \qquad \text{(III.11)}$$

This guarantees the conservation of total kinetic energy, insofar as vertical advection is concerned. The finite difference expression for the kinetic energy in a vertical column is

$$\frac{\pi}{g} \sum_{k=1}^{K}{}' \tfrac{1}{2} \mathbb{V}_k^2 \Delta\sigma_k \ . \tag{III.12}$$

$\sum_{k=1}^{K}{}'$ is the summation for odd k.

5. <u>The pressure gradient force</u>.

We wish to maintain the property of the vertically integrated pressure gradient force discussed in Chapter II, Sec. 2. For this purpose, it is convenient to start from the form given by (I.22)'. We write $\nabla_\sigma(\pi\Phi) - \frac{\partial}{\partial\sigma}(\Phi\sigma)\nabla\pi$ as

$$\nabla(\pi\Phi_k) - \frac{1}{\Delta\sigma_k}(\hat{\Phi}_{k+1}\sigma_{k+1} - \hat{\Phi}_{k-1}\sigma_{k-1})\nabla\pi \ . \tag{III.13}$$

The symbol ^ is a reminder that the variable is at an even level. The analog to (II.4) is

$$-\frac{1}{g} \sum_{k=1}^{K}{}' \text{(III.13)}\ \Delta\sigma_k = -\frac{1}{g}\left[\nabla\left(\sum_{k=1}^{K}{}' \pi(\Phi_k - \hat{\Phi}_S)\Delta\sigma_k\right) + \pi\nabla\hat{\Phi}_S\right]. \tag{III.14}$$

In this way the integral property is maintained.

(III.13) is equivalent to

$$\pi\left[\nabla\Phi_k + \frac{1}{\pi}\left\{\Phi_k - \frac{1}{\Delta\sigma_k}(\hat{\Phi}_{k+1}\sigma_{k+1} - \hat{\Phi}_{k-1}\sigma_{k-1})\right\}\nabla\pi\right]. \tag{III.15}$$

Let

$$\pi(\sigma\alpha)_k \equiv \Phi_k - \frac{1}{\Delta\sigma_k}(\hat{\Phi}_{k+1}\sigma_{k+1} - \hat{\Phi}_{k-1}\sigma_{k-1}) \ . \tag{III.16}$$

This is an analog to (I.10). However, at this stage (III.16) is only the definition of the symbol $(\sigma\alpha)_k$, rather than the hydrostatic equation, because the dependency of $(\sigma\alpha)_k$ on temperature, pressure and σ is not yet specified. Using (III.16), the quantity inside the bracket of (III.15) can be written as

$$\nabla\Phi_k + (\sigma\alpha)_k \nabla\pi \ , \tag{III.17}$$

which is an analog of $\nabla_\sigma \Phi + \sigma\alpha\nabla\pi$ at level k.

6. Kinetic energy generation.

To obtain the kinetic energy generation in finite difference form, we follow the procedure used in deriving (II.7).

$$- \pi \mathbb{V}_k \cdot \left[\nabla\Phi_k + (\sigma\alpha)_k \nabla\pi\right] =$$

$$= - \nabla\cdot(\pi \mathbb{V}_k \Phi_k) - \Phi_k \left[\frac{\partial\pi}{\partial t} + \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1} - \pi\dot{\sigma}_{k-1})\right] - \pi(\sigma\alpha)_k \mathbb{V}_k \cdot \nabla\pi$$

$$= - \nabla\cdot(\pi \mathbb{V}_k \Phi_k) - \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1}\hat{\Phi}_{k+1} - \pi\dot{\sigma}_{k-1}\hat{\Phi}_{k-1}) - \Phi_k \frac{\partial\pi}{\partial t}$$

$$+ \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}(\hat{\Phi}_{k+1} - \Phi_k) + \pi\dot{\sigma}_{k-1}(\Phi_k - \hat{\Phi}_{k-1})\right] - \pi(\sigma\alpha)_k \mathbb{V}_k \cdot \nabla\pi$$

$$= - \nabla\cdot(\pi \mathbb{V}_k \Phi_k) - \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1}\hat{\Phi}_{k+1} - \pi\dot{\sigma}_{k-1}\hat{\Phi}_{k-1}) - (\Phi_k - \pi(\sigma\alpha)_k)\frac{\partial\pi}{\partial t}$$

$$- \pi\left[(\sigma\alpha)_k\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)\pi - \frac{1}{\pi\Delta\sigma_k}\left\{\pi\dot{\sigma}_{k+1}(\hat{\Phi}_{k+1} - \Phi_k) + \pi\dot{\sigma}_{k-1}(\Phi_k - \hat{\Phi}_{k-1})\right\}\right]$$

$$= - \nabla\cdot(\pi \mathbb{V}_k \Phi_k) - \frac{1}{\Delta\sigma_k}\left[\left(\pi\dot{\sigma}_{k+1} + \sigma_{k+1}\frac{\partial\pi}{\partial t}\right)\hat{\Phi}_{k+1} - \left(\pi\dot{\sigma}_{k-1} + \sigma_{k-1}\frac{\partial\pi}{\partial t}\right)\hat{\Phi}_{k-1}\right]$$

$$- \pi(\omega\alpha)_k \ . \tag{III.18}$$

Here $(\omega\alpha)_k$ is defined by

$$(\omega\alpha)_k \equiv (\sigma\alpha)_k \left(\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla\right)\pi - \frac{1}{\pi\Delta\sigma_k}\left\{\pi\dot{\sigma}_{k+1}(\hat{\Phi}_{k+1} - \Phi_k) + \pi\dot{\sigma}_{k-1}(\Phi_k - \hat{\Phi}_{k-1})\right\} . \quad (III.19)$$

At this stage, (III.19) is the definition of the symbol $(\omega\alpha)_k$.

From a finite difference scheme for the first law of thermodynamics, we will determine a form for $(\omega\alpha)_k$. Then, by comparing it with (III.19), we will determine a form of $(\sigma\alpha)_k$ and then a finite difference expression for the hydrostatic equation.

7. <u>The first law of thermodynamics</u>.

As an analog to (I.24)", we use the form given by (III.5) with $A = \theta$.

$$\frac{\partial}{\partial t}(\pi\theta_k) + \nabla\cdot(\pi\mathbb{V}_k\theta_k) + \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1}\hat{\theta}_{k+1} - \pi\dot{\sigma}_{k-1}\hat{\theta}_{k-1}) = \frac{\pi}{c_p}\left(\frac{\theta}{T}\right)_k Q_k \qquad (III.20)$$

By using

$$\hat{\theta}_{k+1} = \tfrac{1}{2}(\theta_k + \theta_{k+2}) , \qquad (III.21)$$

we have a scheme which conserves

$$\int \sum_{k=1}^{K}{}' \pi\theta_k \Delta\sigma_k \, dS \quad \text{and} \quad \int \sum_{k=1}^{K}{}' \pi\theta_k^2 \Delta\sigma_k \, dS$$

under an adiabatic process. Here $\int dS$ is the area integral over the entire globe.

As in (III.6), (III.20) is equivalent to

$$\pi\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)\theta_k + \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}(\hat{\theta}_{k+1} - \theta_k) + \pi\dot{\sigma}_{k-1}(\theta_k - \hat{\theta}_{k-1})\right] = \frac{\pi}{c_p}\left(\frac{\theta}{T}\right)_k Q_k . \quad (III.22)$$

We define θ_k by

$$\theta_k = T_k / p_k^{\varkappa} , \qquad (III.23)$$

where $p_k = p_T + \pi\sigma_k$. (III.22) x $p_k^{\varkappa}$ gives

$$\pi\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)T_k - \pi\theta_k\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)p_k^{\varkappa}$$

$$+\frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}(p_k^{\varkappa}\hat{\theta}_{k+1} - T_k) + \pi\dot{\sigma}_{k-1}(T_k - p_k^{\varkappa}\hat{\theta}_{k-1})\right] = \frac{\pi}{c_p}Q_k ,$$

from which we obtain

$$\pi\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)c_p T_k + \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}c_p(\hat{T}_{k+1} - T_k) + \pi\dot{\sigma}_{k-1}c_p(T_k - \hat{T}_{k-1})\right]$$

$$= \pi\frac{RT_k}{p_k}\sigma_k\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)\pi$$

$$+ \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}c_p(\hat{T}_{k+1} - p_k^{\varkappa}\hat{\theta}_{k+1}) + \pi\dot{\sigma}_{k-1}c_p(p_k^{\varkappa}\hat{\theta}_{k-1} - \hat{T}_{k-1})\right]$$

$$+ \pi Q_k . \tag{III.24}$$

The left hand side of (III.24) may be written, in the flux form, as

$$\frac{\partial}{\partial t}(\pi c_p T_k) + \nabla\cdot(\pi\mathbb{V}_k c_p T_k) + \frac{c_p}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1}\hat{T}_{k+1} - \pi\dot{\sigma}_{k-1}\hat{T}_{k-1}) . \tag{III.25}$$

8. Total energy conservation and the hydrostatic equation.

In order that the total energy be conserved under an adiabatic, frictionless process, the right-hand side of (III.24), except for πQ_k, must agree with $\pi(\omega\alpha)_k$, where $(\omega\alpha)_k$ is defined by (III.19). First we must require

$$(\sigma\alpha)_k = \sigma_k\frac{RT_k}{p_k} . \tag{III.26}$$

Because $(\sigma\alpha)_k$ is defined by (III.16), we have

$$\Phi_k - \frac{1}{\Delta\sigma_k}(\hat{\Phi}_{k+1}\sigma_{k+1} - \hat{\Phi}_{k-1}\sigma_{k-1}) = \pi\sigma_k\frac{RT_k}{p_k} . \tag{III.27}$$

This is a form of the hydrostatic equation which corresponds to (I.10). We must also require that

$$c_p \hat{T}_{k+1} - p_k^{\varkappa} c_p \hat{\theta}_{k+1} = \Phi_k - \hat{\Phi}_{k+1} \ , \tag{III.28}$$

and

$$p_k^{\varkappa} c_p \hat{\theta}_{k-1} - c_p \hat{T}_{k-1} = \hat{\Phi}_{k-1} - \Phi_k \ . \tag{III.28)'$$

Rearranging the terms,

$$(c_p \hat{T}_{k+1} + \hat{\Phi}_{k+1}) - (c_p T_k + \Phi_k) = p_k^{\varkappa} c_p (\hat{\theta}_{k+1} - \theta_k) \ , \tag{III.29}$$

and

$$(c_p T_k + \Phi_k) - (c_p \hat{T}_{k-1} + \hat{\Phi}_{k-1}) = p_k^{\varkappa} c_p (\theta_k - \hat{\theta}_{k-1}) \ . \tag{III.29)'$$

(III.29) and (III.29)' are analogues of another form of the hydrostatic equation, (I.12). $\hat{\theta}_{k+1}$ (and therefore $\hat{\theta}_{k-1}$) is given by (III.21).

Replacing k, in (III.29)', by k+2 and adding it to (III.29), we obtain

$$(c_p T_{k+2} + \Phi_{k+2}) - (c_p T_k + \Phi_k) = c_p \tfrac{1}{2}(p_{k+2}^{\varkappa} + p_k^{\varkappa})(\theta_{k+2} - \theta_k) \ ,$$

or

$$\Phi_{k+2} - \Phi_k = - c_p (p_{k+2}^{\varkappa} - p_k^{\varkappa}) \frac{\theta_{k+2} + \theta_k}{2} \ . \tag{III.30}$$

(III.30) is an analogue of (I.11). (III.30) shows that Φ_k(at odd levels) is related to the neighboring levels as if θ is linear in $p^{\varkappa}$ between the levels.

(III.30) is used for computing Φ at the odd levels. To do so, we need to know Φ at a single (odd) level, say, $k = K$. We use (III.27) for this purpose. From (III.27), we obtain

$$\sum_{k=1}^{K}{}' \Phi_k \Delta\sigma_k - \Phi_S = \sum_{k=1}^{K}{}' \pi \sigma_k \frac{R T_k}{p_k} \Delta\sigma_k \ , \tag{III.31}$$

which is an analogue to (II.5). We can write $\sum_{k=1}^{K}{}' \Phi_k \Delta\sigma_k$ as

$$\sum_{k=1}^{K}{}' \Phi_k \Delta\sigma_k = \sum_{k=1}^{K}{}' \Phi_k (\sigma_{k+1} - \sigma_{k-1})$$

$$= \Phi_K + \sum_{k=1}^{K-2}{}' \sigma_{k+1} (\Phi_k - \Phi_{k+2}) \; . \qquad \text{(III.32)}$$

Using (III.30) in (III.32) gives

$$\sum_{k=1}^{K}{}' \Phi_k \Delta\sigma = \Phi_K + \sum_{k=1}^{K-2}{}' \sigma_{k+1}\, c_p (p_{k+2}^{\varkappa} - p_k^{\varkappa}) \tfrac{1}{2} (\theta_{k+2} + \theta_k)$$

$$= \Phi_K + \sum_{k=1}^{K-2}{}' \frac{c_p}{2} \left[\sigma_{k+1} (p_{k+2}^{\varkappa} - p_k^{\varkappa}) + \sigma_{k-1} (p_k^{\varkappa} - p_{k-2}^{\varkappa}) \right] \theta_k$$

$$+ \frac{c_p}{2} \sigma_{K-1} (p_K^{\varkappa} - p_{K-2}^{\varkappa}) \theta_K$$

$$= \Phi_K + \sum_{k=1}^{K-2}{}' \frac{c_p}{2} \left[\sigma_{k+1} \left(\left(\frac{p_{k+2}}{p_k}\right)^{\varkappa} - 1 \right) + \sigma_{k-1} \left(1 - \left(\frac{p_{k-2}}{p_k}\right)^{\varkappa} \right) \right] T_k$$

$$+ \frac{c_p}{2} \sigma_{K-1} \left(1 - \left(\frac{p_{K-2}}{p_K}\right)^{\varkappa} \right) T_K \; . \qquad \text{(III.33)}$$

Substituting (III.33) into (III.31), we obtain

$$\Phi_K = \Phi_S + \sum_{k=1}^{K}{}' \left[\pi \sigma_k \frac{R}{p_k} \Delta\sigma_k - c_p (\sigma_{k+1} \beta_k + \sigma_{k-1} \alpha_k) \right] T_k \; , \qquad \text{(III.34)}$$

where

$$\beta_k \equiv \begin{cases} \tfrac{1}{2} \left(\left(\frac{p_{k+2}}{p_k} \right)^{\varkappa} - 1 \right) & \text{for } k \le K-2, \\ 0 & \text{for } k = K, \end{cases}$$

$$\alpha_k \equiv \begin{cases} 0 \text{ or any value} & \text{for } k = 1, \\ \tfrac{1}{2} \left(1 - \left(\frac{p_{k-2}}{p_k} \right)^{\varkappa} \right) & \text{for } k \ge 3 \; . \end{cases} \qquad \text{(III.35)}$$

9. Summary of sections 5-8.

We have now constructed a vertical difference scheme which maintains the property of the vertically integrated horizontal pressure gradient force, total energy conservation under adiabatic and frictionless processes, and conservations of θ and θ^2, integrated over the entire mass, under adiabatic processes.

Pressure gradient force (per unit mass):

From (III.17) and (III.26),

$$-\left[\nabla\Phi_k+\sigma_k\frac{RT_k}{p_k}\nabla\pi\right], \qquad \text{(III.36)}$$

where $p_k = p_T + \pi\sigma_k$.

The hydrostatic equation:

(III.34)

$$\Phi_K = \Phi_S + \sum_{k=1}^{K}{}' \left[\pi\sigma_k\frac{R}{p_k}\Delta\sigma_k - c_p(\sigma_{k+1}\beta_k+\sigma_{k-1}\alpha_k)\right]T_k. \qquad \text{(III.37}$$

(III.35)

$$\beta_k \equiv \begin{cases} \frac{1}{2}\left(\left(\frac{p_{k+2}}{p_k}\right)^{\varkappa}-1\right) & \text{for } k \le K-2, \\ 0 & \text{for } k = K, \end{cases}$$

$$\alpha_k \equiv \begin{cases} 0 \text{ or any value} & \text{for } k = 1, \\ \frac{1}{2}\left(1-\left(\frac{p_{k+2}}{p_k}\right)^{\varkappa}\right) & \text{for } k \ge 3. \end{cases} \qquad \text{(III.38)}$$

And from (III.30) and (III.38)

$$\Phi_k - \Phi_{k+2} = c_p\,(\alpha_{k+2}T_{k+2} + \beta_k T_k). \qquad \text{(III.39)}$$

The first law of thermodynamics

Using (III.25) for the left hand side of (III.24), rearranging, the terms, and dividing by c_p,

$$\frac{\partial}{\partial t}(\pi T_k) + \nabla\cdot(\pi \mathbb{V}_k T_k) + \frac{1}{\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}(p_k^{\varkappa}\hat{\theta}_{k+1}) - \pi\dot{\sigma}_{k-1}(p_k^{\varkappa}\hat{\theta}_{k-1})\right]$$

$$= \pi\frac{\varkappa T_k}{p_k}\sigma_k\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)\pi + \pi Q_k/c_p\,, \qquad \text{(III.40)}$$

which corresponds to $(I.26)/c_p$.

From (III.21),

$$p_k^{\varkappa}\hat{\theta}_{k+1} = \tfrac{1}{2}\left(T_k + \left(\frac{p_k}{p_{k+2}}\right)^{\varkappa} T_{k+2}\right),$$

$$p_k^{\varkappa}\hat{\theta}_{k-1} = \tfrac{1}{2}\left(\left(\frac{p_k}{p_{k-2}}\right)^{\varkappa} T_{k-2} + T_k\right). \qquad \text{(III.41)}$$

Among the results given in Secs. 5-8, (III.36-41) are all that we need for the main computations. However, for computing moist convection, we also need (III.29) and (III.29)', which, with (III.21), can be written as

$$(c_p\hat{T}_{k+1} + \hat{\Phi}_{k+1}) - (c_p T_k + \Phi_k) = \frac{c_p}{2}p_k^{\varkappa}(\theta_{k+2} - \theta_k)\,, \qquad \text{(III.42)}$$

and

$$(c_p T_k + \Phi_k) - (c_p\hat{T}_{k-1} + \hat{\Phi}_{k-1}) = \frac{c_p}{2}p_k^{\varkappa}(\theta_k - \theta_{k-2})\,. \qquad \text{(III.42)'}$$

For diagnostic analyses of the results of the computations, such as the vertical transfers of energy and momentum, we may need to know $\hat{\Phi}_{k+1}$ and $\hat{T}_{k+1}$ separately. $\hat{\Phi}_{k+1}$ is obtained by summing (III.27) from k = 1 to k. $\hat{T}_{k+1}$ is then obtained from (III.42).

IV. VERTICAL DIFFERENCING, PART B

1. The flux form for the water vapor equation.

The flux form (III.5) applied to the water vapor equation is

$$\frac{\partial}{\partial t}(\pi q_k) + \nabla\cdot(\pi \mathbb{V}_k q_k) + \frac{1}{\Delta\sigma_k}(\pi\dot{\sigma}_{k+1} q_{k+1} - \pi\dot{\sigma}_{k-1} q_{k-1}) = \pi(-C + E)_k . \qquad \text{(IV.1)}$$

This form guarantees the conservation of total water vapor when there are no water vapor sources and sinks. (IV.1) is equivalent to

$$\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)q_k + \frac{1}{\pi\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}(q_{k+1} - q_k) + \pi\dot{\sigma}_{k-1}(q_k - q_{k-1})\right] = (-C + E)_k , \qquad \text{(IV.2)}$$

but we must choose q at even levels properly.

2. Moist adiabatic process--continuous case.

Consider, first, the moist adiabatic process in the continuous atmosphere. Let the air be saturated and remain saturated, and let there be no heating other than the heat of condensation.

Let $q^*(T,p)$ be the saturation mixing ratio. Then, the water vapor equation is

$$\frac{d}{dt} q^* = -C , \qquad \text{(IV.3)}$$

when condensation is occurring. The first law of thermodynamics is

$$\frac{d}{dt} c_p T = \omega\alpha + LC , \qquad \text{(IV.4)}$$

where L is the heat of condensation per unit mass.

From (IV.3), we obtain

$$\left(\frac{\partial q^*}{\partial T}\right)_p \frac{dT}{dt} + \left(\frac{\partial q^*}{\partial p}\right)_T \omega = - C \ . \qquad \text{(IV.5)}$$

From (IV.4) and (IV.5) we have

$$C = - \frac{\omega}{1 + \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p} \left[\left(\frac{\partial q^*}{\partial p}\right)_T + \frac{\alpha}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p \right] . \qquad \text{(IV.6)}$$

Substituting (IV.6) into (IV.4), we obtain

$$\frac{dT}{dt} = \omega \left(\frac{\partial T}{\partial p}\right)_m \ , \qquad \text{(IV.7)}$$

where

$$\left(\frac{\partial T}{\partial p}\right)_m \equiv \frac{\frac{\alpha}{c_p} - \frac{L}{c_p}\left(\frac{\partial q^*}{\partial p}\right)_T}{1 + \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p} \ , \qquad \text{(IV.8)}$$

or

$$\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_p T = \omega \left[\left(\frac{\partial T}{\partial p}\right)_m - \frac{\partial T}{\partial p} \right] . \qquad \text{(IV.9)}$$

Here $\partial/\partial p$ without the suffix is the derivative under constant horizontal coordinates and constant time.

The corresponding equation with the σ-coordinate can be readily obtained by using the following relations in (IV.9):

$$\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_p = \left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_\sigma - \frac{\sigma}{\pi}\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)\pi \frac{\partial}{\partial \sigma} \ , \qquad \text{(from I.15)}$$

and

$$\omega = \sigma \left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)\pi + \pi\dot{\sigma} \ . \qquad \text{(I.5)}$$

Then,

$$\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_\sigma T = \left(\frac{\partial T}{\partial p}\right)_m \sigma \left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)\pi + \pi\dot{\sigma}\left[\left(\frac{\partial T}{\partial p}\right)_m - \frac{\partial T}{\partial p} \right] , \qquad \text{(IV.10)}$$

where

$$\frac{\partial}{\partial p} = \frac{1}{\pi}\frac{\partial}{\partial \sigma} . \qquad (I.8)$$

Using the relation

$$\frac{\partial q^*}{\partial p} = \left(\frac{\partial q^*}{\partial p}\right)_T + \left(\frac{\partial q^*}{\partial T}\right)_p \frac{\partial T}{\partial p} , \qquad (IV.11)$$

(IV.8) and (IV.11) give

$$\begin{aligned}\left(\frac{\partial T}{\partial p}\right)_m - \frac{\partial T}{\partial p} &= \frac{1}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p}\left[\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} - \frac{L}{c_p}\frac{\partial q^*}{\partial p}\right] \\ &= \frac{1}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p}\left[-p^{\varkappa}\frac{\partial \theta}{\partial p} - \frac{L}{c_p}\frac{\partial q^*}{\partial p}\right] .\end{aligned} \qquad (IV.12)$$

From (IV.10) and (IV.12), we obtain

$$\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_\sigma T = \left(\frac{\partial T}{\partial p}\right)_m \sigma\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_\sigma \pi - \frac{p^{\varkappa}\frac{\partial \theta}{\partial p} + \frac{L}{c_p}\frac{\partial q^*}{\partial p}}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p}\pi\dot{\sigma} . \qquad (IV.13)$$

Equation (I.12) is

$$c_p\, p^{\varkappa}\left(\frac{\partial \theta}{\partial p}\right) = \frac{\partial}{\partial p}(c_p T + \Phi) .$$

Defining h and h^* by

$$h \equiv c_p T + \Phi + Lq , \qquad h^* \equiv c_p T + \Phi + Lq^* , \qquad (IV.14)$$

we obtain

$$p^{\varkappa}\frac{\partial \theta}{\partial p} + \frac{L}{c_p}\frac{\partial q^*}{\partial p} = \frac{1}{c_p}\frac{\partial h^*}{\partial p} .$$

Finally, (IV. 13) may be written as

$$\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_\sigma T = \left(\frac{\partial T}{\partial p}\right)_m \sigma\left(\frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right)_\sigma \pi - \frac{\frac{\partial h^*}{\partial p}}{c_p + L\left(\frac{\partial q^*}{\partial T}\right)_p}\pi\dot{\sigma} . \qquad (IV.15)$$

3. Moist adiabatic process--discrete case.

Let $q_k^* \equiv q^*(T_k, p_k)$. When level k is saturated and remains saturated, (IV.2) may be rewritten as

$$(\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla) q_k^* + \frac{1}{\pi \Delta\sigma_k} \left[\pi\dot{\sigma}_{k+1}(q_{k+1} - q_k^*) + \pi\dot{\sigma}_{k-1}(q_k^* - q_{k-1}) \right] = - C \,, \quad \text{(IV.16)}$$

when condensation is occurring.

The first law of thermodynamics is, from (III.24),

$$(\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla) T_k + \frac{1}{\pi \Delta\sigma_k} \left[\pi\dot{\sigma}_{k+1}(p_k^{\varkappa}\theta_{k+1} - T_k) + \pi\dot{\sigma}_{k-1}(T_k - p_k^{\varkappa}\theta_{k-1}) \right]$$

$$= \frac{1}{c_p} \alpha_k \sigma_k (\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla)\pi + \frac{L}{c_p} C \,, \quad \text{(IV.17)}$$

where $\alpha_k \equiv \frac{RT_k}{p_k}$.

From (IV.16), we obtain

$$(\frac{\partial q^*}{\partial T})_{p_k} (\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla) T_k + (\frac{\partial q^*}{\partial p})_{T_k} \sigma_k (\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla)\pi$$

$$+ \frac{1}{\pi\Delta\sigma_k} \left[\pi\dot{\sigma}_{k+1}(q_{k+1} - q_k^*) + \pi\dot{\sigma}_{k-1}(q_k^* - q_{k-1}) \right] = - C \,, \quad \text{(IV.18)}$$

where

$$(\frac{\partial q^*}{\partial T})_{p_k} \equiv (\frac{\partial q_k^*}{\partial T_k})_{p_k} \,, \quad (\frac{\partial q^*}{\partial p})_{T_k} \equiv (\frac{\partial q_k^*}{\partial p_k})_{T_k} .$$

(IV.17) and (IV.18) give

$$C = - \frac{1}{1 + \frac{L}{c_p}(\frac{\partial q^*}{\partial T})_{p_k}} \Bigg[\left\{ (\frac{\partial q^*}{\partial p})_{T_k} + \frac{\alpha_k}{c_p} (\frac{\partial q^*}{\partial T})_{p_k} \right\} \sigma_k (\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla)\pi$$

$$- (\frac{\partial q^*}{\partial T})_{p_k} \frac{p_k^{\varkappa}}{\pi\Delta\sigma_k} \left\{ \pi\dot{\sigma}_{k+1}(\theta_{k+1} - \theta_k) + \pi\dot{\sigma}_{k-1}(\theta_k - \theta_{k-1}) \right\}$$

$$+ \frac{1}{\pi\Delta\sigma_k} \left\{ \pi\dot{\sigma}_{k+1}(q_{k+1} - q_k^*) + \pi\dot{\sigma}_{k-1}(q_k^* - q_{k-1}) \right\} \Bigg] . \quad \text{(IV.19)}$$

Define $(\partial T/\partial p)_{mk}$ by

$$\left(\frac{\partial T}{\partial p}\right)_{mk} \equiv \frac{\frac{\alpha_k}{c_p} - \frac{L}{c_p}\left(\frac{\partial q^*}{\partial p}\right)_{Tk}}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_{pk}} .$$

Substituting (IV.19) into (IV.17), we have

$$\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)T_k = \left(\frac{\partial T}{\partial p}\right)_{mk} \sigma_k\left(\frac{\partial}{\partial t} + \mathbb{V}_k\cdot\nabla\right)\pi$$

$$- \frac{1}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_{pk}} \frac{1}{\pi\Delta\sigma_k}\left[\pi\dot{\sigma}_{k+1}\left(p_k^{\varkappa}\theta_{k+1} + \frac{L}{c_p}q_{k+1} - p_k^{\varkappa}\theta_k - \frac{L}{c_p}q_k^*\right)\right.$$

$$\left. + \pi\dot{\sigma}_{k-1}\left(p_k^{\varkappa}\theta_k + \frac{L}{c_p}q_k^* - p_k^{\varkappa}\theta_{k-1} - \frac{L}{c_p}q_{k-1}\right)\right], \qquad \text{(IV.20)}$$

(IV.20) is an analogue of (IV.13).

The coefficient of $\pi\dot{\sigma}_{k+1}$ in (IV.20) is

$$p_k^{\varkappa}(\theta_{k+1}-\theta_k) + \frac{L}{c_p}(q_{k+1}-q_k^*)$$

$$= p_k^{\varkappa}\tfrac{1}{2}(\theta_{k+2}-\theta_k) + \frac{L}{c_p}(q_{k+1}-q_k^*)$$

$$= \frac{1}{c_p}\left[(c_pT_{k+1} + \Phi_{k+1} + Lq_{k+1}) - (c_pT_k + \Phi_k + Lq_k^*)\right]$$

$$\equiv \frac{1}{c_p}(h_{k+1} - h_k^*) . \qquad \text{(IV.21)}$$

Here (III.21) and (III.42) were used. Similarly, the coefficient of $\pi\dot{\sigma}_{k-1}$ in (IV.20) is

$$\frac{1}{c_p}(h_k^* - h_{k-1}) . \qquad \text{(IV.21)'}$$

Then, from (IV.20) we obtain

$$\left(\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla\right) T_k = \left(\frac{\partial T}{\partial p}\right)_{mk} \sigma_k \left(\frac{\partial}{\partial t} + \mathbb{V}_k \cdot \nabla\right) \pi$$

$$- \frac{1}{c_p + L\left(\frac{\partial q^*}{\partial T}\right)_{pk}} \frac{1}{\pi \Delta \sigma_k} \left[\pi\dot{\sigma}_{k+1} (h_{k+1} - h_k^*) \right.$$

$$\left. + \pi\dot{\sigma}_{k-1} (h_k^* - h_{k-1}) \right] . \qquad \text{(IV.22)}$$

(IV.22) is an analogue of (IV.15).

Let us suppose that $\pi\dot{\sigma}_{k+1}$ is negative and $h_{k+1} > h_k^*$. From (IV.21)

$$\frac{L}{c_p} (q_{k+1} - q_k^*) > p_k^{\varkappa} (\theta_k - \theta_{k+1}) \ . \qquad \text{(IV.23)}$$

From (III.21), $\theta_k - \theta_{k+1} = \frac{1}{2}(\theta_k - \theta_{k+2})$, and this is maintained positive or zero (dry adiabatically stable or neutral) by the dry convective adjustment which will be described later. Therefore, $q_{k+1} > q_k^*$ when $h_{k+1} > h_k^*$. The coefficient of $\pi\dot{\sigma}_{k+1}$ in the bracket of (IV.19) is $\left(\frac{\partial q^*}{\partial T}\right)_{pk} p_k^{\varkappa}(\theta_k - \theta_{k+1}) + (q_{k+1} - q_k^*)$, except for a constant factor. This coefficient is positive. Consequently, the negative $\pi\dot{\sigma}_{k+1}$ makes a positive contribution to the condensation. (This is not true when $q_{k+1} < q_k^*$. In that case, the negative $\pi\dot{\sigma}$ pumps drier air up from below.) From (IV.22) we see that the negative $\pi\dot{\sigma}_{k+1}$ has a warming effect for $h_{k+1} > h_k^*$, which leads to a moist convective instability. This instability may occur even when no conditional instability exists between the odd levels, which carry the temperatures and the mixing ratios. Then the instability is a result of a poor choice of q at the even levels and is a kind of computational instability.

We may call this "conditional instability of a computational kind (CICK)".

Similarly, when $\pi\dot{\sigma}_{k-1}$ is negative, $h_{k-1} \geq h_k^*$ is required for stability. Therefore, when condensation is taking place at the level k, we must choose q_{k+1} and q_{k-1} which give

$$\begin{aligned} h_{k+1} &\leq h_k^* \qquad \text{when } \pi\dot{\sigma}_{k+1} < 0, \\ h_{k-1} &\geq h_k^* \qquad \text{when } \pi\dot{\sigma}_{k-1} < 0. \end{aligned} \tag{IV.24}$$

There are three situations for a particular even level which has negative $\pi\dot{\sigma}$ as shown below.

(a)	condensation	k		
	―――――――	k+1	$h_{k+1} \leq h_k^*$,	
	no condensation	k+2		
(b)	no condensation	k		
	―――――――	k+1	$h_{k+1} \geq h_{k+2}^*$,	(IV.25)
	condensation	k+2		
(c)	condensation	k		
	―――――――	k+1	$h_{k+2}^* \leq h_{k+1} \leq h_k^*$.	
	condensation	k+2		

4. Choice of q at the even levels.

The difficulties of vertical differencing of the water vapor equation are not limited to the saturated case.

For the potential temperature, the arithmetic average, $\theta_{k+1} = \frac{1}{2}(\theta_k + \theta_{k+2})$, was used. The integral constraint on θ^2 was maintained and, together with the conservation of the average θ, this resulted in a conservation of the variance of θ. But the arithmetic average, $q_{k+1} = \frac{1}{2}(q_k + q_{k+2})$, is not a comparable good choice, because, unlike θ, the variance of q is so great that conservations of only its lower moments are not effective constraints on its frequency distribution.

Moreover, the arithmetic average does not guarantee that q remains positive or zero. For example, if $q_k = 0$, $q_{k+1} > 0$ and $\pi\dot{\sigma}_{k+1} > 0$, then the downward current removes a positive amount from zero.

The upstream scheme

$$q_{k+1} = q_k \qquad \text{for} \quad \pi\dot{\sigma}_{k+1} > 0$$

$$q_{k+1} = q_{k+2} \qquad \text{for} \quad \pi\dot{\sigma}_{k+1} < 0$$

does not produce this difficulty. But the upstream scheme tends to make q homogeneous in the vertical, and thereby produces an excessive condensation in the upper levels.

At present (December 1971) we are testing various choices of q at the even levels.

V. INTRODUCTION TO HORIZONTAL DIFFERENCING

1. Distribution of variables over the grid points.

We now consider the horizontal grid and the way of distributing the variables over the grid points.

Our governing equations are the primitive equations. Under normal conditions in the atmosphere (low Rossby and Froude numbers), these equations govern two well-separable types of motions. One type is low-frequency, quasi-geostrophic motion; the other is high-frequency gravity-inertia waves. It is known that the energy of locally excited gravity-inertia waves disperses away into a wider space, leaving the slowly changing quasi-geostrophic motion behind. This process is called "geostrophic adjustment".

Consequently, there are two main computational problems in simulating large-scale motions with the primitive equations. One computational problem is the proper simulation of the geostrophic adjustment. The other computational problem is the simulation of the slowly changing quasi-geostrophic (and, therefore, quasi-nondivergent) motion after it has been established by geostrophic adjustment.

Winninghoff and Arakawa examined the geostrophic adjustment process with various finite difference schemes and found that it depends on how the variables are distributed over the grid points. The following discussion is taken from their paper.*

* Winninghoff, F. J. and A. Arakawa, 1972: "Dispersion Properties of Gravity-Inertia Waves in Space-Centered Difference Schemes and Their Effect on the Geostrophic Adjustment Process. (In Preparation).

Dispersion properties of gravity-inertia waves in space-centered difference schemes and their effect on the geostrophic adjustment process.

Consider the simplest fluid in which geostrophic adjustment can take place: namely, an incompressible, homogeneous, non-viscous, hydrostatic, rotating fluid with a flat bottom and a free top surface.

The basic equations which govern such a fluid are:

$$\frac{du}{dt} - fv + g\frac{\partial h}{\partial x} = 0 \,, \tag{1.1}$$

$$\frac{dv}{dt} + fu + g\frac{\partial h}{\partial y} = 0 \,, \tag{1.2}$$

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \,, \tag{1.3}$$

where t is time, x and y are the horizontal coordinates, u and v are the velocity components respectively in the x and y directions, h is the depth of the fluid, f is a constant coriolis parameter, and g is gravity. The individual time rate of change is

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \,. \tag{1.4}$$

In most of this study we consider the problem with a linearized version of these equations. The linearization is done by replacing d/dt by $\partial/\partial t$; and replacing h as the factor on $(\partial u/\partial x + \partial v/\partial y)$ in equation (1.3) by H, the mean value of h. This is justified if the Rossby number is small and the horizontal scale is of the order of the radius of deformation.

We consider five ways of distributing the dependent variables, h, u and v, in a square grid in space, as illustrated in figure 1.

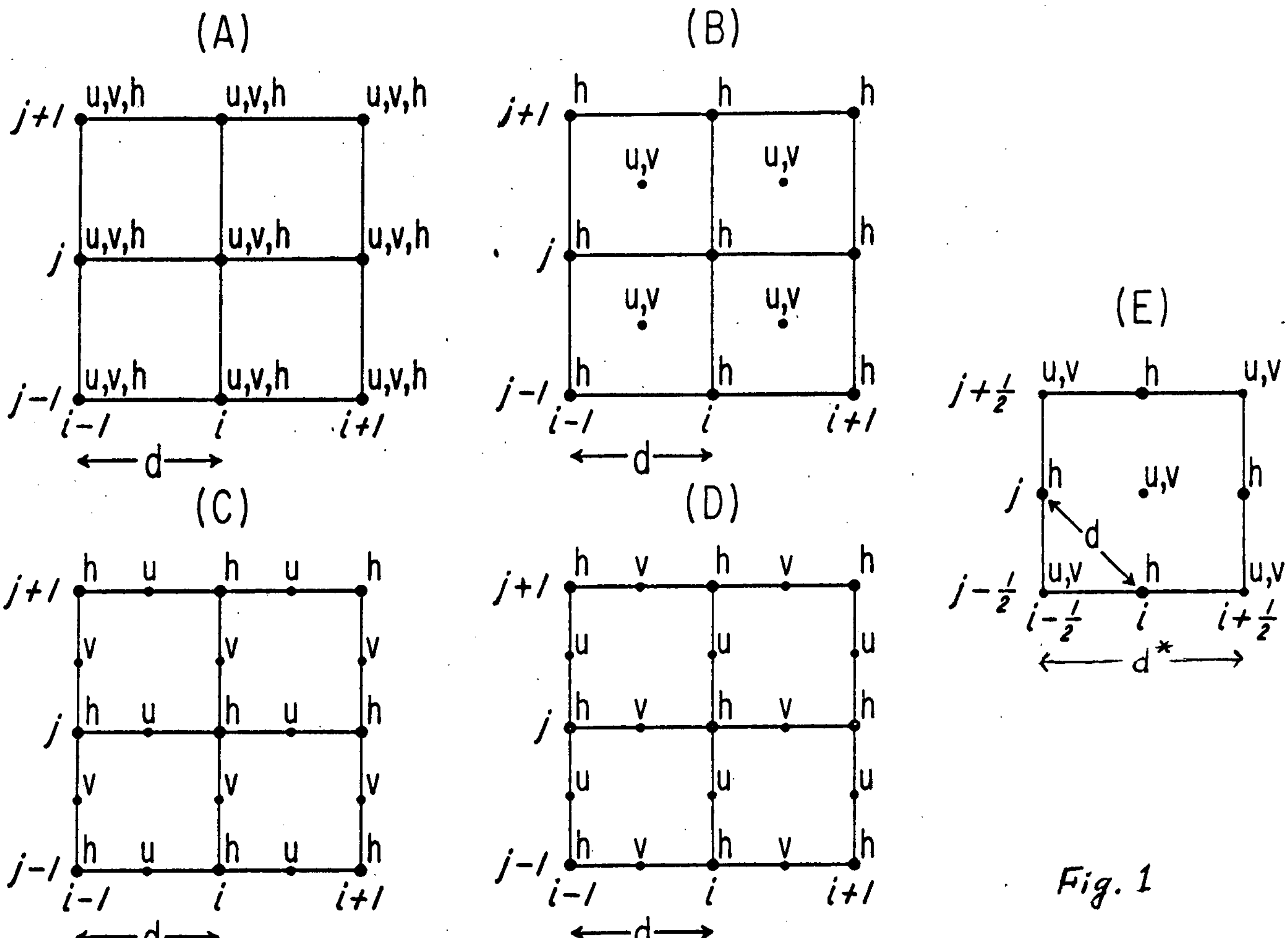


Fig. 1

The space finite difference schemes we use for the linearized equations are the simplest second-order schemes for each of these five ways of distributing the variables. They are:

scheme A,

$$\frac{\partial u}{\partial t} - fv + g\overline{(\delta_x h)}^x = 0 \; , \tag{1.5}$$

$$\frac{\partial v}{\partial t} + fu + g\overline{(\delta_y h)}^y = 0 \; , \tag{1.6}$$

$$\frac{\partial h}{\partial t} + H\left[\overline{(\delta_x u)}^x + \overline{(\delta_y v)}^y\right] = 0 \; , \tag{1.7}$$

scheme B,

$$\frac{\partial u}{\partial t} - fv + g\overline{(\delta_x h)}^y = 0 \; , \tag{1.8}$$

$$\frac{\partial v}{\partial t} + fu + g\overline{(\delta_y h)}^x = 0 \; , \tag{1.9}$$

$$\frac{\partial h}{\partial t} + H\left[\overline{(\delta_x u)}^y + \overline{(\delta_y v)}^x\right] = 0 \; ; \tag{1.10}$$

scheme C,

$$\frac{\partial u}{\partial t} - f\overline{v}^{xy} + g(\delta_x h) = 0 \; , \tag{1.11}$$

$$\frac{\partial v}{\partial t} + f\overline{u}^{xy} + g(\delta_y h) = 0 \; , \tag{1.12}$$

$$\frac{\partial h}{\partial t} + H\left[(\delta_x u) + (\delta_y v)\right] = 0 \; ; \tag{1.13}$$

scheme D,

$$\frac{\partial u}{\partial t} - f\overline{v}^{xy} + g\overline{(\delta_x h)}^{xy} = 0 \; , \tag{1.14}$$

$$\frac{\partial v}{\partial t} + f\overline{u}^{xy} + g\overline{(\delta_y h)}^{xy} = 0 \; ; \tag{1.15}$$

$$\frac{\partial h}{\partial t} + H\left[\overline{(\delta_x u)}^{xy} + \overline{(\delta_y v)}^{xy}\right] = 0 \; ; \tag{1.16}$$

scheme E,

(1.17) $$\frac{\partial u}{\partial t} - fv + g(\delta_x h) = 0 ,$$

(1.18) $$\frac{\partial v}{\partial t} + fu + g(\delta_y h) = 0 ,$$

(1.19) $$\frac{\partial h}{\partial t} + H\left[(\delta_x u) + (\delta_y v)\right] = 0 ;$$

where we define,

(1.20) $$(\delta_x \alpha)_{ij} \equiv \frac{1}{d^*}\left(\alpha_{i+\frac{1}{2},\,j} - \alpha_{i-\frac{1}{2},\,j}\right) ,$$

(1.21) $$(\overline{\alpha}^x)_{ij} \equiv \frac{1}{2}\left(\alpha_{i+\frac{1}{2},\,j} + \alpha_{i-\frac{1}{2},\,j}\right) ,$$

where i and j are the indices of the grid points in the x and y directions. For the schemes A through D, d^* is the grid size d, as shown in figure 1. For scheme E, d^* equals $\sqrt{2}$ times d, as shown in the figure. This choice of d for scheme E will make scheme E have the same number of grid points as the other schemes in a given two-dimensional domain. $(\delta_y \alpha)_{ij}$ and $(\overline{\alpha}^y)_{ij}$ are defined in a similar manner, but with respect to the y direction. Finally,

(1.22) $$(\overline{\alpha}^{xy})_{ij} \equiv (\overline{\overline{\alpha}^{x}}^{y})_{ij} .$$

In this study, all analyses with the linearized equations leave the time-change terms in differential form. The reason for doing this is that unless an implicit scheme is used, we must choose a time interval short enough to satisfy the Courant-Friedrichs-Lewy type condition for linear computational stability of the wave which has the largest possible phase speed. For the primitive equations of atmospheric motion, this is the Lamb wave. In that case, the time interval is adequately small for all other waves, including internal gravity waves, and we can then ignore the time truncation error in the first approximation.

We consider, first, one-dimensional linear equations. The equations we use are:

(2.1) $$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0 ,$$

(2.2) $$\frac{\partial v}{\partial t} + fu = 0 ,$$

(2.3) $$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0 .$$

Eliminating v and h, we obtain,

(2.4) $$\frac{\partial^2 u}{\partial t^2} + f^2 u - gH\frac{\partial^2 u}{\partial x^2} = 0 .$$

If we assume that the solution has a form proportional to $e^{i(kx-\nu t)}$, then the frequency ν is given by

(2.5) $$\left(\frac{\nu}{f}\right)^2 = 1 + gH\left(\frac{k}{f}\right)^2 ,$$

where k is the wave number in the x direction.

Next, we examine the effect of the space truncation error on the frequency. In this one-dimensional case, the space distributions of the dependent variables, for the schemes A through E, are shown in figure 2.

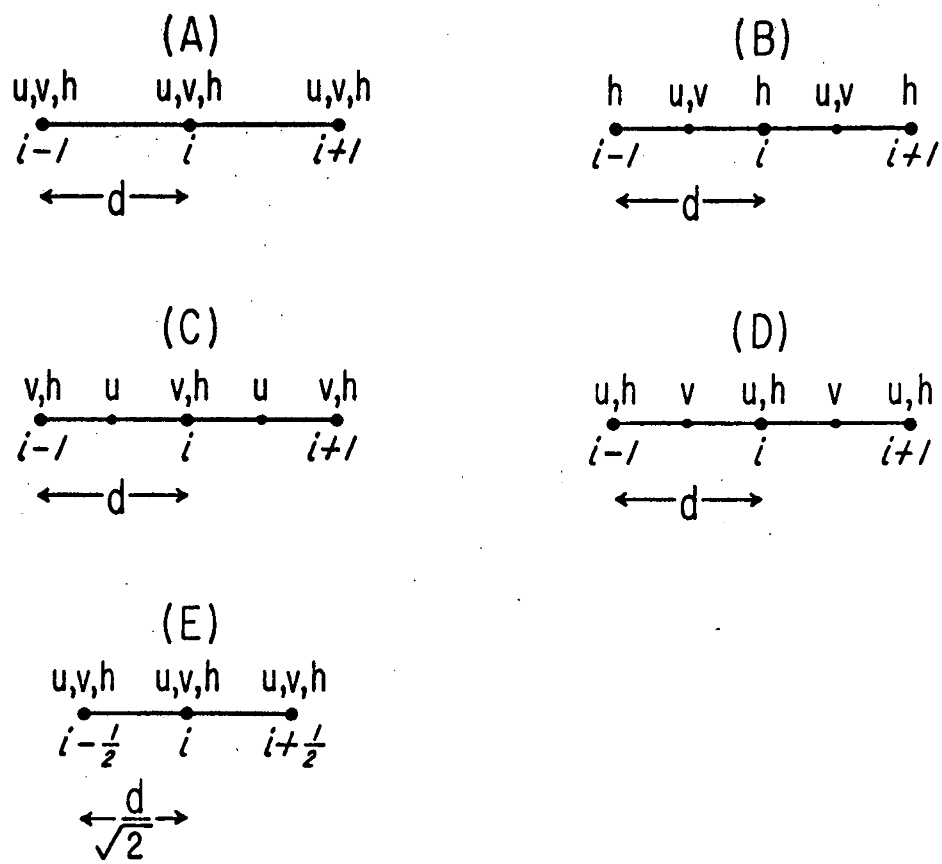


Fig. 2

The forms of the space difference schemes are:

scheme A,

(2.6) $$\frac{\partial u}{\partial t} - fv + g\overline{(\delta_x h)}^x = 0 ,$$

(2.7) $$\frac{\partial v}{\partial t} + fu = 0 ,$$

(2.8) $$\frac{\partial h}{\partial t} + H\overline{(\delta_x v)}^x = 0 ;$$

scheme B,

(2.9) $$\frac{\partial u}{\partial t} - fv + g(\delta_x h) = 0 ,$$

(2.10) $$\frac{\partial v}{\partial t} + fu = 0 ,$$

(2.11) $$\frac{\partial h}{\partial t} + H(\delta_x u) = 0 ;$$

scheme C,

(2.12) $$\frac{\partial u}{\partial t} - f\overline{v}^x + g(\delta_x h) = 0$$

(2.13) $$\frac{\partial v}{\partial t} + f\overline{u}^x = 0 ,$$

(2.14) $$\frac{\partial h}{\partial t} + H(\delta_x u) = 0 ;$$

scheme D,

(2.15) $$\frac{\partial u}{\partial t} - f\overline{v}^x + g\overline{(\delta_x h)}^x = 0 ;$$

(2.16) $$\frac{\partial v}{\partial t} + f\overline{u}^x = 0 ,$$

(2.17) $$\frac{\partial h}{\partial t} + H\overline{(\delta_x u)}^x = 0 ;$$

scheme E,

$$\frac{\partial u}{\partial t} - fv + g(\delta_x h) = 0 \; . \tag{2.18}$$

$$\frac{\partial v}{\partial t} + fu = 0 \; . \tag{2.19}$$

$$\frac{\partial h}{\partial t} + H(\delta_x u) = 0 \; . \tag{2.20}$$

In this one-dimensional case, scheme E is equivalent to scheme A, but with a smaller grid size.

For each of the schemes, we obtain the following frequencies

$$\text{A:} \quad \left(\frac{\nu}{f}\right)^2 = 1 + \left(\frac{\lambda}{d}\right)^2 \sin^2 kd \quad , \tag{2.21}$$

$$\text{B:} \quad \left(\frac{\nu}{f}\right)^2 = 1 + 4\left(\frac{\lambda}{d}\right)^2 \sin^2\left(\frac{kd}{2}\right) \; , \tag{2.22}$$

$$\text{C:} \quad \left(\frac{\nu}{f}\right)^2 = \cos^2\left(\frac{kd}{2}\right) + 4\left(\frac{\lambda}{d}\right)^2 \sin^2\left(\frac{kd}{2}\right) \quad , \tag{2.23}$$

$$\text{D:} \quad \left(\frac{\nu}{f}\right)^2 = \cos^2\left(\frac{kd}{2}\right) + \left(\frac{\lambda}{d}\right)^2 \sin^2 (kd) \quad , \tag{2.24}$$

$$\text{E:} \quad \left(\frac{\nu}{f}\right)^2 = 1 + 2\left(\frac{\lambda}{d}\right)^2 \sin^2\left(\frac{kd}{\sqrt{2}}\right) \; . \tag{2.25}$$

$\sqrt{gH}$ is the speed of the gravity wave, which is the theoretical maximum group velocity of the gravity-inertia waves given by equation (2.5). The radius of deformation, λ, is defined by $\sqrt{gH}/f$. The non-dimensional frequency, ν/f, depends on two parameters, kd and λ/d.

From these frequencies for the gravity-inertia waves in each scheme, we can see their dispersion properties. From (2.5) for the continuous case, we see that the frequency of gravity-inertia waves is a monotonically increasing function of the wave number k, unless the radius of deformation, $\lambda \equiv \sqrt{gH}/f$, is zero. Then the group velocity, $\partial\nu/\partial k$, is not zero unless $\lambda = 0$. This non-zero group velocity is very important for the geostrophic adjustment process.

The wave length of the shortest resolvable wave is 2d. The corresponding wave number, k_{max}, is π/d. Therefore, in examining equations (2.21)-(2.25), it is sufficient to consider the range $0 < kd < \pi$.

Scheme A: The frequency reaches its maximum at $kd = \pi/2$. This means that the group velocity at $kd = \pi/2$ is zero. When gravity-inertia waves at about this wave number are excited somewhere in the domain (by non-linearity, heating, etc.), the wave energy stays there. Moreover, a wave with $kd = \pi$ behaves like a pure inertia oscillation.

Scheme B: It produces a monotonically increasing frequency for non-zero λ, in the range $0 < kd < \pi$.

Scheme C: The frequency monotonically increases for $\frac{\lambda}{d} > \frac{1}{2}$ and monotonically decreases for $\lambda/d < \frac{1}{2}$. For $\frac{\lambda}{d} = \frac{1}{2}$, the group velocity is zero for all k.

Scheme D: The frequency reaches a maximum at $(\frac{\lambda}{d})^2 \cos kd = \frac{1}{4}$. Moreover, $kd = \pi$ is a stationary wave.

Scheme E: The frequency reaches a maximum at $kd = \frac{\pi}{\sqrt{2}}$.

These results for the one-dimensional case show that scheme B is the most satisfactory. However, when λ/d is sufficiently larger than 1/2, scheme C is as good as scheme B. For internal gravity waves, it is the reduced gravity which matters. When scheme C with a coarse grid is used for an atmosphere which has a relatively weak stratification, λ/d may not be sufficiently larger than 1/2 and, therefore, geostrophic adjustment is poorly simulated by scheme C. This comes from the fact that the averaging of the coriolis force, in scheme C, makes the shortest resolvable motion behave as if it were on a non-rotating frame.

Figure 3 shows the case for $\frac{\lambda}{d} = 2$ (for which scheme C is also good).

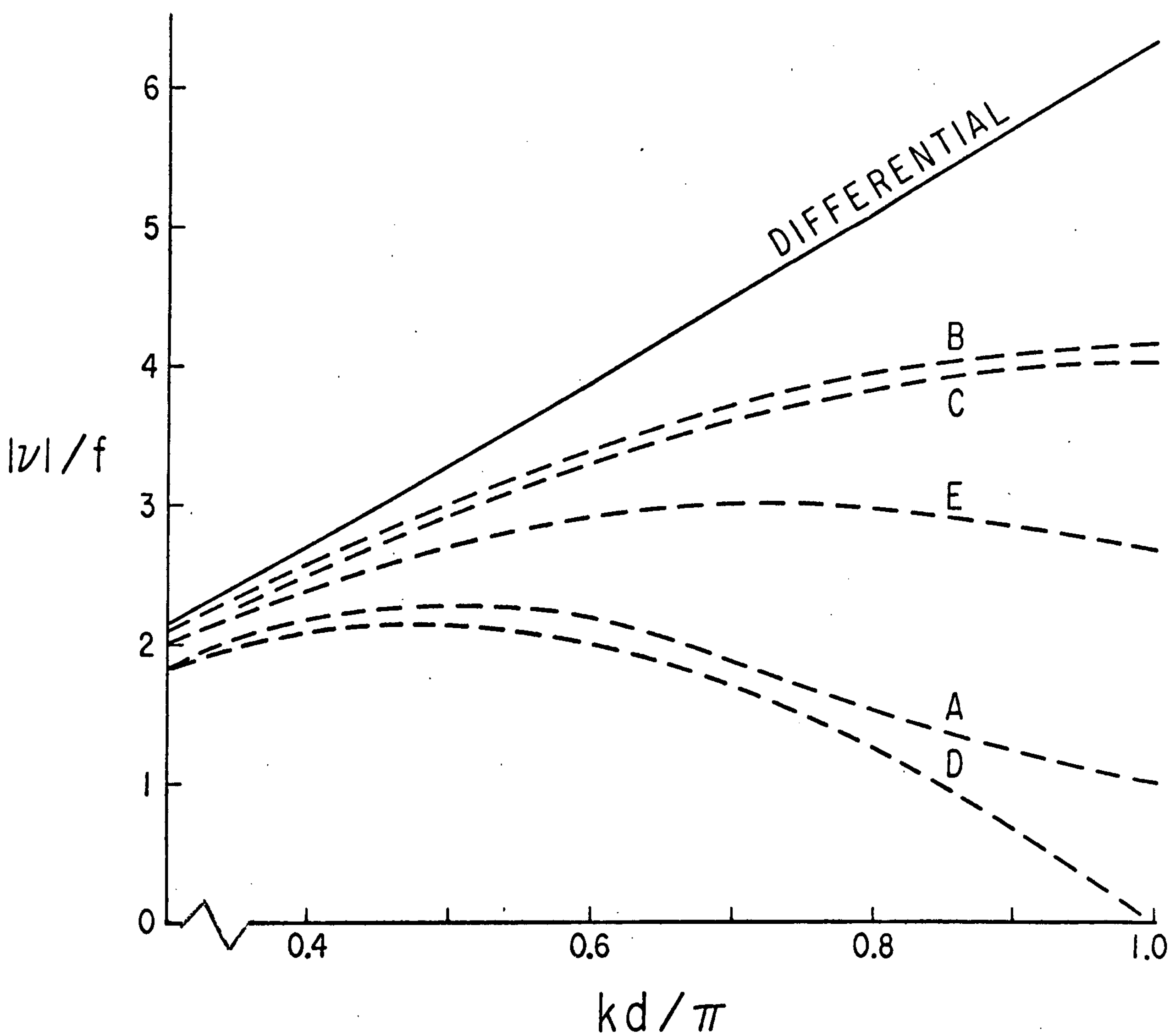


Fig. 3

Cahn* gave the solution of an initial value problem for which (2.1) to (2.3) are the governing equations. At the initial time, he let h = constant, $u = 0$ and $v = V_o$ in the domain from $x = -a$ to $x = a$, and $v = 0$ outside of this domain.

We obtain the solution, $u(x,t)$, for this same initial condition, by writing the solution in the Fourier integral form:

$$u(x,t) = \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{\infty} e^{ikx} u^*(k,t)dk \ , \tag{2.26}$$

$$u^*(k,t) = \int_{-\infty}^{\infty} e^{-ikx} u(x,t)dx \ , \tag{2.27}$$

where k is the wave number in the x direction, and $u^*(k,t)$ satisfies

$$\frac{\partial^2 u^*(k,t)}{\partial t^2} + (f^2 + k^2 gH)u^*(k,t) = 0 \ . \tag{2.28}$$

Equation (2.28) has the general solution,

$$u^*(k,t) = A(k)\cos(\nu t) + B(k)\sin(\nu t) \ , \tag{2.29}$$

where,

$$\nu^2 = f^2(1+\lambda^2 k^2) \ . \tag{2.30}$$

To determine $A(k)$ and $B(k)$, we apply equations (2.29) and (2.27) at $t = 0$.

Then,

$$A(k) = u^*(k,o) = \int_{-\infty}^{\infty} e^{-ikx} u(x,o)dx = 0 \ . \tag{2.31}$$

---

*Cahn, A., "An Investigation of the Free Oscillations of a Simple Current System", Journal of Meteorology, Vol. 2, No. 2, June 1945, pp. 113-119.

Moreover, equations (2.29) and (2.27) give

$$\frac{\partial u^*(k,t)}{\partial t} = \nu\left[-A(k)\sin(\nu t) + B(k)\cos(\nu t)\right], \tag{2.32}$$

and

$$\frac{\partial u^*(k,t)}{\partial t} = \int_{-\infty}^{\infty} e^{-ikx}\,\frac{\partial u(x,t)}{\partial t}\,dx\,. \tag{2.33}$$

Applying equations (2.32) and (2.33) at $t = 0$, we obtain

$$B(k) = \frac{1}{\nu}\left(\frac{\partial u^*(k,t)}{\partial t}\right)_{t=o} = \frac{1}{\nu}\int_{-\infty}^{\infty} e^{-ikx}\left(\frac{\partial u(x,t)}{\partial t}\right)_{t=o} dx\,. \tag{2.34}$$

From the initial conditions and equation (2.1), we have

$$\left(\frac{\partial u(x,t)}{\partial t}\right)_{t=o} = \begin{matrix} fV_o & \text{for } |x| \le a \\ 0 & \text{for } |x| > a\,. \end{matrix} \tag{2.35}$$

Therefore, from equation (2.34),

$$B(k) = \frac{1}{\nu}\int_{-a}^{a} e^{-ikx}\,fV_o\,dx = -\frac{fV_o\,e^{-ikx}}{\nu\quad ik}\Bigg|_{x=-a}^{x=a} \tag{2.36}$$

$$= \frac{2fV_o}{k\nu}\sin(ak)\,.$$

Finally, we obtain

$$u(x,t) = \frac{faV_o}{\pi}\,\mathrm{Re}\int_{-\infty}^{\infty}\frac{\sin(ak)}{ak}\,\frac{\sin(\nu t)}{\nu}\,e^{ikx}\,dk\,, \tag{2.37}$$

or

$$u(x,t) = \frac{2faV_o}{\pi}\int_{-\infty}^{\infty}\frac{\sin(ak)}{ak}\,\frac{\sin(\nu t)}{\nu}\cos kx\,dk\,. \tag{2.38}$$

We can also write down the equations corresponding to equation (2.38), for each of the finite difference schemes, A through E. These are identical to equation (2.38), except for the frequency ν, which is given by the expressions (2.21) to (2.25) for each scheme.

We evaluated the integral (2.38) numerically for the differential case and for each of the finite difference schemes. We used Simpson's rule, with 600 intervals in k from 0 to π/a, to compute u at $x = 0$ for various values of t. Then we obtained h, at $x = a$, from the equation of continuity. These solutions, for a constant x, were calculated for values of t up to 40 hours, at 15 minute intervals with $f = 10^{-4} \sec^{-1}$. Since equation (2.38) also enables us to evaluate the solution for constant t over a range of x, this was also done for each of the cases.

Some results of the calculations, with $a/d = 1$ and $\lambda/d = 2$, are shown in figures 4 and 5. Figure 4 shows the <u>time</u> variation of h for the differential case, which approximates the solution obtained by Cahn, and for each of the difference schemes, at $x = a$. Figure 5 gives the <u>space</u> variation of h in the differential case and for each of the schemes, at $t = 80$ hours.

As we expected, scheme B (together with scheme C in this case where $\lambda/d > \frac{1}{2}$) better simulates the geostrophic adjustment than the other schemes. However, even scheme B has a difficulty in the two-dimensional case. Figure 6 shows $|\nu|/f$ as a function of kd/π and $\ell d/\pi$, for scheme B, where k and ℓ are the wave numbers in the x and y directions. Again, $\lambda/d = 2$.

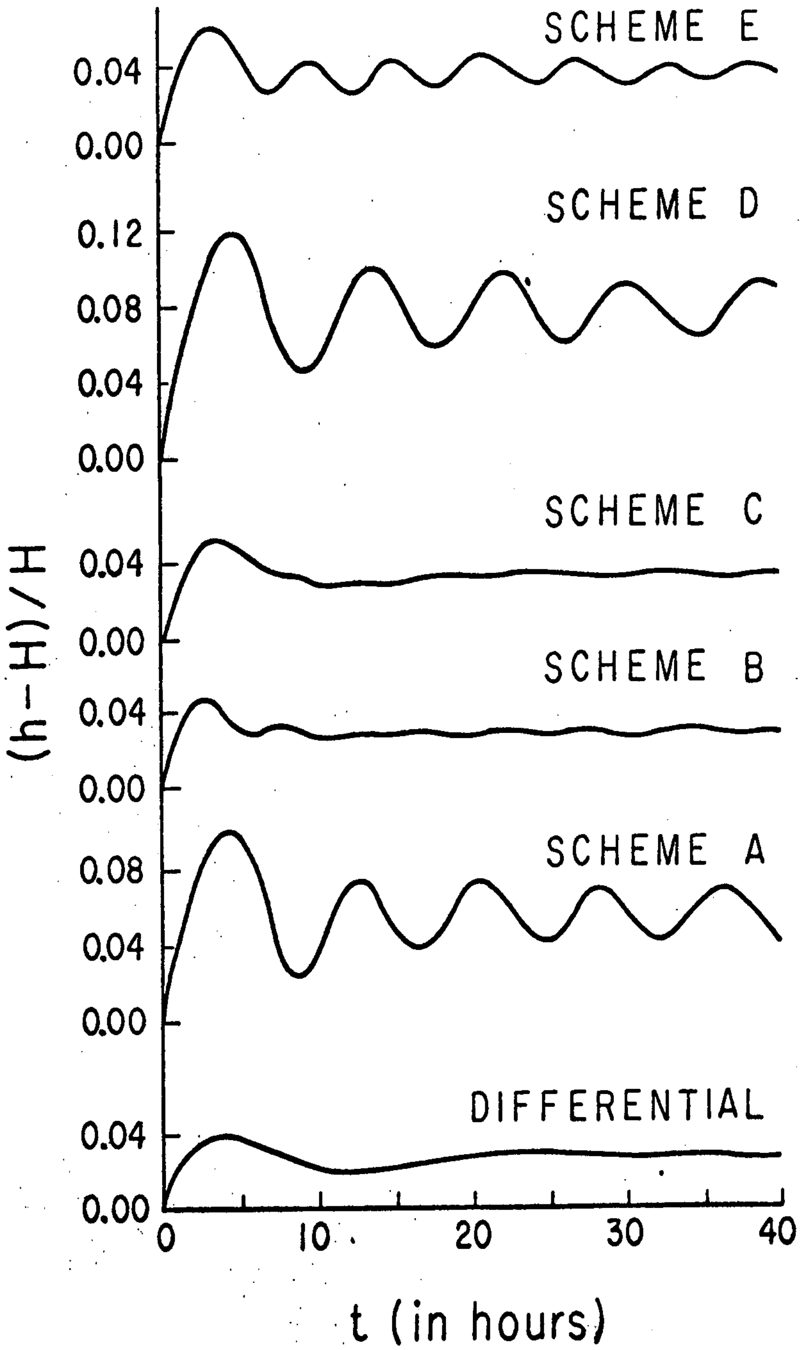

SCHEME E
SCHEME D
SCHEME C
SCHEME B
SCHEME A
DIFFERENTIAL
(h−H)/H
0.04
0.00
0.12
0.08
0.04
0.00
0.04
0.00
0.04
0.00
0.08
0.04
0.00
0.04
0.00
0
10
20
30
40
t (in hours)

Fig. 4

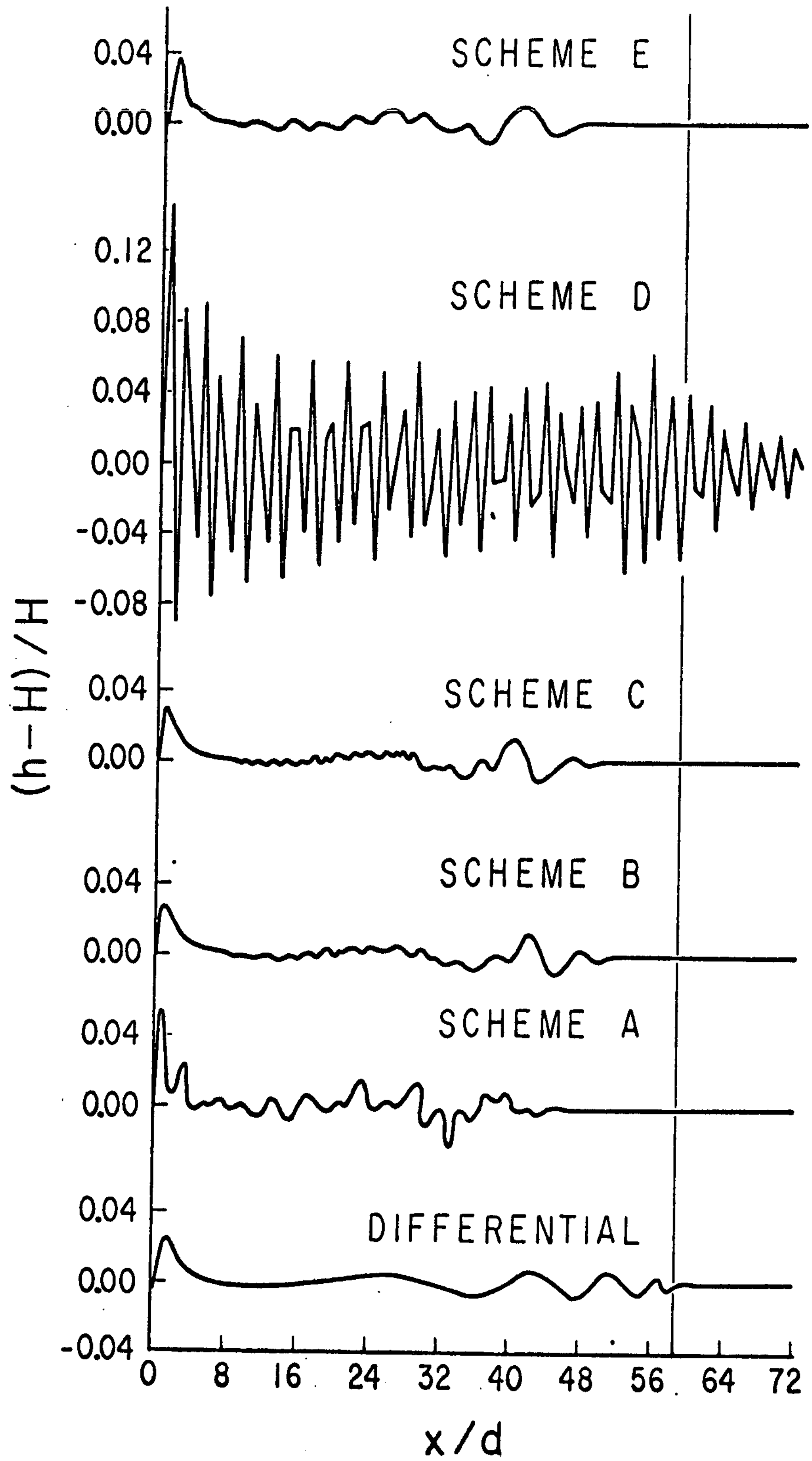

SCHEME E
SCHEME D
SCHEME C
SCHEME B
SCHEME A
DIFFERENTIAL
(h−H)/H
0.04
0.00
0.12
0.08
0.04
0.00
-0.04
-0.08
0.04
0.00
0.04
0.00
0.04
0.00
0.04
0.00
-0.04
0
8
16
24
32
40
48
56
64
72
x/d

Fig. 5

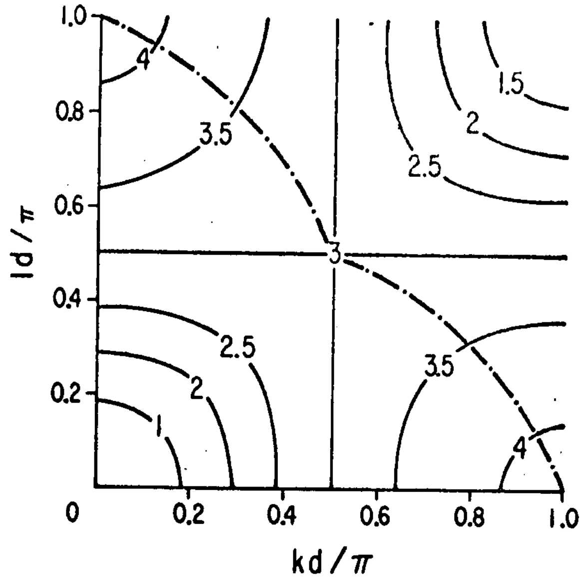


Fig. 6

It thus appears that all space-centered schemes have some deficiency. Because these deficiencies are due to the existence of false low frequencies for high wave number motions, we cannot expect that any form of time-differencing will avoid these deficiencies as long as we use space-centered schemes at each time step. As shown, in Chapter 8, we overcome these deficiencies by using a specially designed "time-alternating space-uncentered" difference scheme (TASU).

2. Two-dimensional nondivergent flow.

We now must consider the simulation of the slowly changing quasi-geostrophic (and, therefore, quasi-nondivergent) motion after it is established by the geostrophic adjustment process.

We look, first, at the flow which is purely two-dimensional and nondivergent. The following is extracted from the review paper by Arakawa* (1970).

The principal computational problems can be illustrated with the vorticity equation,

(13) $$\partial\zeta/\partial t = J(\zeta,\psi) ,$$

where

(14) $$J(\zeta,\psi) = (\partial\zeta/\partial x)(\partial\psi/\partial y) - (\partial\zeta/\partial y)(\partial\psi/\partial x) ,$$

(15) $$\zeta = \nabla^2\psi ,$$

and ψ is the streamfunction.

There are the following integral constraints, among others, on the Jacobian:

(16) $$\overline{J(\zeta,\psi)} = 0 ,$$

(17) $$\overline{\zeta J(\zeta,\psi)} = 0 ,$$

(18) $$\overline{\psi J(\zeta,\psi)} = 0 ,$$

where the bar denotes the average over the domain, along the boundary of which ψ is constant. From these integral constraints, we can see that the mean vorticity, $\overline{\zeta}$, the mean square vorticity, $\overline{\zeta^2}$, and the mean kinetic energy, $\frac{1}{2}\overline{(\nabla\psi)^2}$, are conserved with time. Conservation of these quantities during the advection process poses important constraints on the statistical properties of two-dimensional incompressible flow. In particular, the average wave number, k, defined as

(19) $$k^2 = \overline{(\nabla^2\psi)^2}/\overline{(\nabla\psi)^2} ,$$

is conserved with time, so that no systematic cascade of energy into shorter waves can occur.

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*Arakawa, A., "Numerical Simulation of Large-Scale Atmospheric Motions", Numerical Solution of Field Problems in Continuum Physics, (Proceedings of a Symposium in Applied Mathematics, Durham, N.C., 1968), Vol. 2, SIAM-AMS Proceedings; edited by G. Birkhoff and S. Varga, American Mathematical Society, Providence, R.I., 1970, pp. 24-40.

If we wish to simulate the statistical properties numerically, we must use a finite difference scheme which approximately conserves these quadratic quantities. Avoiding computational instability, in the nonlinear sense, is necessary but not sufficient for this purpose. Two examples of stable schemes which have a false energy cascade into shorter waves will be shown later.

It should be noted that if we apply Equation (13) to a one dimensional problem, the nonlinearity will be lost. Therefore, the tests of a finite difference scheme for incompressible flow must be made with multi-dimensional problems.

The finite difference approximation for Equation (13) may be written, in a relatively general form, as

(20) $$\zeta_{jk}^{n+1} - \zeta_{jk}^{n} = \Delta t \mathbb{J}_{jk}(\zeta^*, \psi^*) ,$$

where $\zeta_{jk}^n = (\nabla_{jk}^2 \psi)^n$ is a finite difference approximation of $\zeta = \nabla^2 \psi$ at the grid point $x = j\Delta x$, $y = k\Delta y$, and $t = n\Delta t$. Δx and Δy are the grid intervals, and Δt is the time interval. ∇_{jk}^2 and $\mathbb{J}_{jk}$ are finite difference approximations for the operators ∇^2 and J at the grid point $x = j\Delta x$, $y = k\Delta y$. Hereafter, the suffixes j, k will be omitted unless they are necessary. There are a number of schemes corresponding to different choices of ζ^* and ψ^*. For example, ζ^* may be a linear combination of ζ^n and ζ^{n+1}, such as,

(21) $$\zeta^* = \tfrac{1}{2}(\zeta^n + \zeta^{n+1}) ,$$

which is an implicit scheme of the Crank-Nicolson type. Or ζ^* may be a provisional value of ζ, prediction by

(22) $$\zeta^* = S\zeta^n + \alpha \Delta t \mathbb{J}^*(\zeta^n, \psi^n) ,$$

where S and α are 1, as in the Matsuno scheme, or S is a smoothing operator and $\alpha = \frac{1}{2}$, as in the two-step Lax-Wendroff scheme. Here $\mathbb{J}^*$ is not necessarily the same as $\mathbb{J}$. Or ζ^* may be extrapolated from ζ^{n-1} and ζ^n, as in the second order Adams-Bashforth scheme; that is,

(23) $$\zeta^* = \frac{3}{2}\zeta^n - \frac{1}{2}\zeta^{n-1} .$$

The change of mean square vorticity is obtained from (20), as

(25) $$\overline{(\zeta^{n+1})^2} - \overline{(\zeta^n)^2} = \Delta t\overline{(\zeta^{n+1} + \zeta^n)\mathbb{J}(\zeta^*, \psi^*)} ,$$

where the bar denotes the average over all grid points in the domain considered. Equation (25) can be rewritten as

(26) $$\overline{(\zeta^{n+1})^2} - \overline{(\zeta^n)^2} = \overline{(\zeta^{n+1} + \zeta^n - 2\zeta^*)(\zeta^{n+1} - \zeta^n)} + 2\Delta t\overline{\zeta^* \mathbb{J}(\zeta^*, \psi^*)} .$$

To conserve mean square vorticity, we must properly choose ζ^* and the form of $\mathbb{J}$ in such a way that the right-hand side of Equation (26) vanishes. The first term on the right vanishes if ζ^* is chosen as $(\zeta^{n+1} + \zeta^n)/2$. The second term vanishes if the finite difference Jacobian, $\mathbb{J}$, maintains the integral constraint given by (17) on the differential Jacobian, J. Similarly, it can be shown that a properly defined kinetic energy is conserved if ψ^* is chosen as $(\psi^{n+1} + \psi^n)/2$ and $\mathbb{J}$ maintains the integral constraint given by (18).

Consider the following three second order, finite difference Jacobians:

(42) $$\begin{aligned} \mathbb{J}_1 &= \Delta_x\zeta\Delta_y\psi - \Delta_y\zeta\Delta_x\psi , \\ \mathbb{J}_2 &= \Delta_y(\psi\Delta_x\zeta) - \Delta_x(\psi\Delta_y\zeta) , \\ \mathbb{J}_3 &= \Delta_x(\zeta\Delta_y\psi) - \Delta_y(\zeta\Delta_x\psi) , \end{aligned}$$

where $\Delta_x f(x)$ denotes $[f(x+d) - f(x-d)]/2d$. Δ_y is defined similarly with respect to y. It was shown by Arakawa* [1966] that the Jacobian, $\mathbb{J}$, given by

(43) $$\begin{aligned} \mathbb{J} &= \alpha\mathbb{J}_1 + \gamma\mathbb{J}_2 + \beta\mathbb{J}_3 , \\ \alpha &+ \gamma + \beta = 1 , \end{aligned}$$

conserves mean square vorticity if $\alpha = \beta$, and conserves energy if $\alpha = \gamma$. Examples of Jacobians which have the form of (43) are:

(44) $$\begin{aligned} \mathbb{J}_4 &= \tfrac{1}{2}(\mathbb{J}_1 + \mathbb{J}_2) , \\ \mathbb{J}_5 &= \tfrac{1}{2}(\mathbb{J}_2 + \mathbb{J}_3) , \\ \mathbb{J}_6 &= \tfrac{1}{2}(\mathbb{J}_3 + \mathbb{J}_1) , \\ \mathbb{J}_7 &= \tfrac{1}{3}(\mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3) . \end{aligned}$$

*A. Arakawa, 1966: Computational Design for Long-Term Numerical Integration of the Equations of Fluid Motion: Two Dimensional Incompressible Flow. Part I. Journ. Computation Physics, Vol. 1, No. 1, pp. 119-143.

$\mathbb{J}_7$ is the Jacobian proposed by Arakawa [1966]. It conserves both mean square vorticity and energy. $\mathbb{J}_2$ and $\mathbb{J}_6$ conserve mean square vorticity, but not energy. $\mathbb{J}_3$ and $\mathbb{J}_4$ conserve energy, but not mean square vorticity. These five schemes are stable. $\mathbb{J}_1$ does not conserve either quantity, and an analysis similar to the analysis by Phillips* [1959], but with the implicit scheme (21), shows that it is unstable. $\mathbb{J}_5$ also does not conserve either quantity, but experience with numerical tests shows that the instability is very weak, if it exists at all. This is not surprising as $2\mathbb{J}_5 = 3\mathbb{J}_7 - \mathbb{J}_1$. Because $\mathbb{J}_7$ is a quadratic-conserving scheme the time rates of change of the mean quadratic quantities using $\mathbb{J}_5$, for given ζ and ψ, have the opposite sign to the time rates of change of the mean quadratic quantities using $\mathbb{J}_1$.

$\mathbb{J}_7$ is the best second order scheme, because of its formal guarantee for maintaining the integral constraints on the quadratic quantities. $\mathbb{J}_7$ is also just as accurate as any other second order scheme. A further increase of the accuracy can be obtained by going to higher order schemes. The more accurate fourth order scheme, which has the same integral constraints as $\mathbb{J}_7$, was also given by Arakawa [1966].

Numerical tests have been made with the above seven Jacobians. In these tests, the initial condition was given by

(45) $$\psi = \Psi \sin(\pi j/8)(\cos(\pi k/8) + 0.1 \cos(\pi k/4)) ,$$

and Δt chosen such that $\Psi \Delta t/d^2 = 0.7$. The leapfrog scheme was used instead of the implicit scheme. In order to eliminate the gradual separation of the solutions at even and odd time steps, which occurs in the leapfrog scheme, a two-level scheme was inserted once in every 240 time steps. The simplest 5-point Laplacian was used. Figures 7

*N. A. Phillips, 1959: An example of non-linear computational instability. The Atmosphere and the Sea in Motion, Rockefeller Press, N.Y., pp. 501-504.

and 8 show the mean square vorticity and the energy obtained with the seven Jacobians. We observe the expected conservation properties, although the implicit scheme was not used. The energy conserving schemes, $\mathbb{J}_3$ and $\mathbb{J}_4$, show considerable increase of mean square vorticity.

Figure 9 shows the spectral distribution of kinetic energy, obtained by the energy and mean square vorticity conserving scheme $\mathbb{J}_7$ and by the energy conserving scheme $\mathbb{J}_3$, at the end of the calculations. The small arrow shows the wave number for $\sin(\pi j/8) \cos(\pi k/8)$, which contained almost all of the energy at the initial time. Although the total energy was approximately conserved with $\mathbb{J}_3$, there was a considerable spurious energy cascade into the high wave numbers; whereas with $\mathbb{J}_7$ more energy went into a lower wave number than into the higher wave numbers, in agreement with the conservation of the average wave number, as given by Equation (19).

Whether the increase of the mean square vorticity is important in the simulation of large-scale atmospheric motion will depend on the viscosity which is used with the complete equation. A relatively small amount of viscosity may be sufficient to keep the mean square vorticity quasi-constant in time. However, the viscosity will also remove energy; and as a result the average wave number, defined by Equation (19), will falsely increase with time.

On the other hand, the mean square vorticity conserving schemes, $\mathbb{J}_2$ and $\mathbb{J}_6$, approximately conserve energy, in spite of no formal guarantee. This is reasonable, because the mean square vorticity is more sensitive to shorter waves, for which the truncation errors are large. $\mathbb{J}_5$ approximately conserves both quantities, again in spite of no formal guarantees. $\mathbb{J}_5$, like $\mathbb{J}_1$ and $\mathbb{J}_7$, maintains the property of the Jacobian, $J(\zeta,\psi) = - J(\psi,\zeta)$. $\mathbb{J}_7$ conserves both quantities, with only negligible errors arising from the leapfrog scheme.

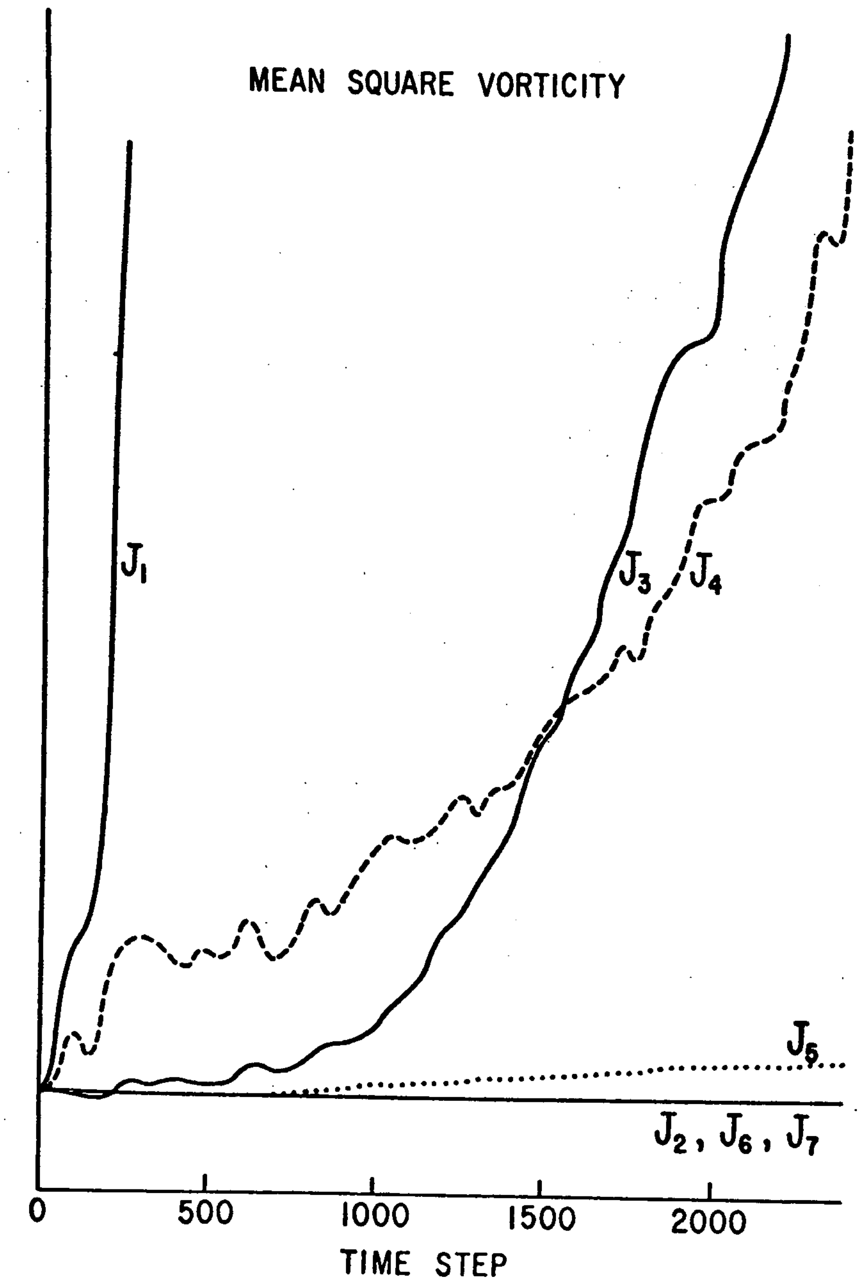

MEAN SQUARE VORTICITY
J_1
J_3
J_4
J_5
J_2, J_6, J_7
0
500
1000
1500
2000
TIME STEP

Fig. 7

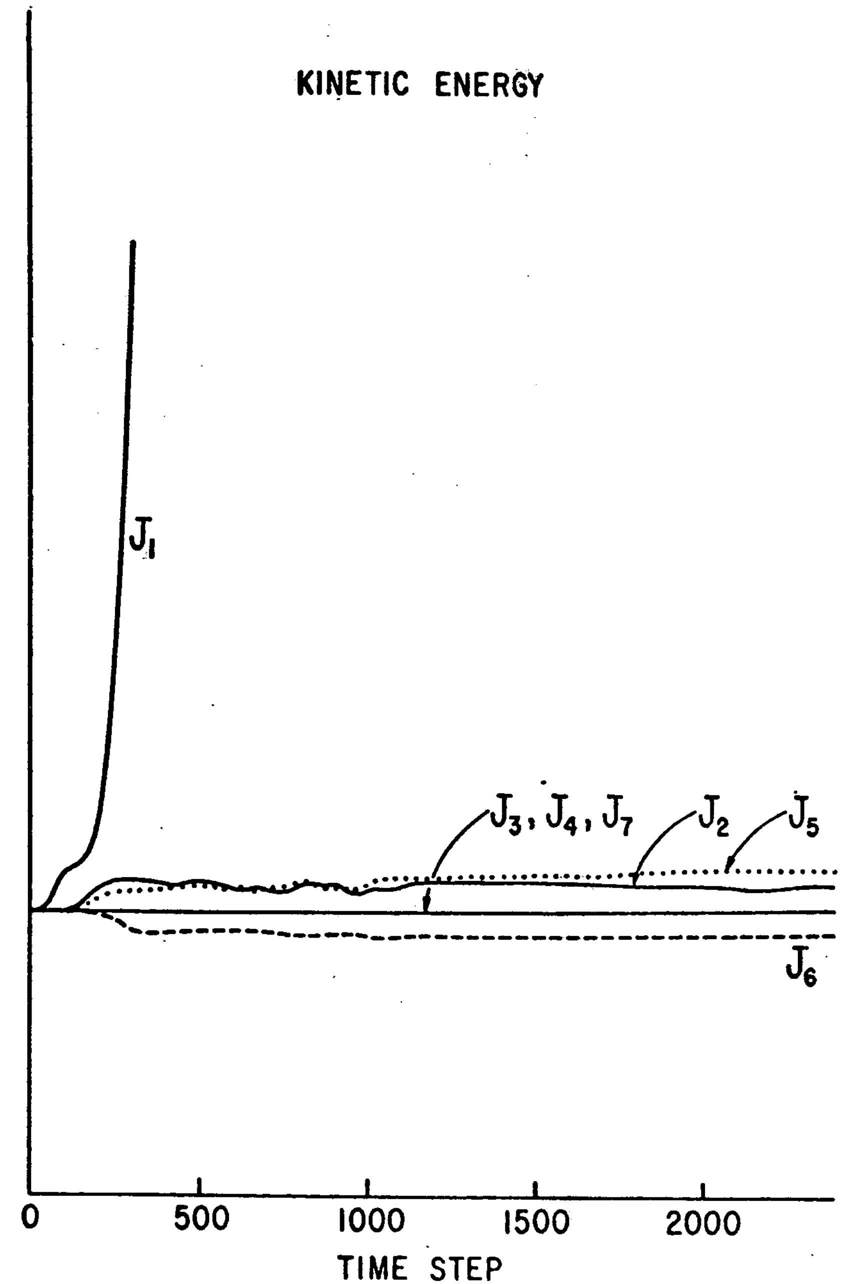
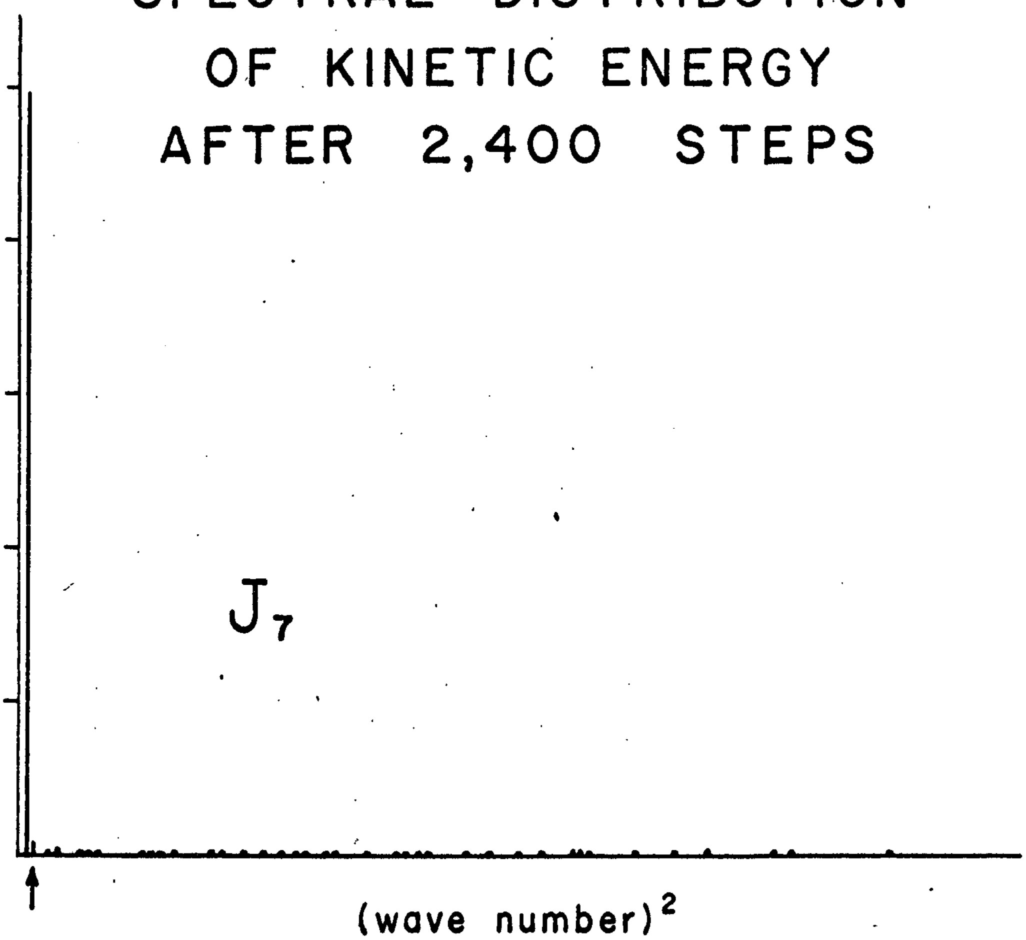

KINETIC ENERGY
J_1
J_3, J_4, J_7
J_2
J_5
J_6
0
500
1000
1500
2000
TIME STEP

Fig. 8

SPECTRAL DISTRIBUTION OF KINETIC ENERGY AFTER 2,400 STEPS



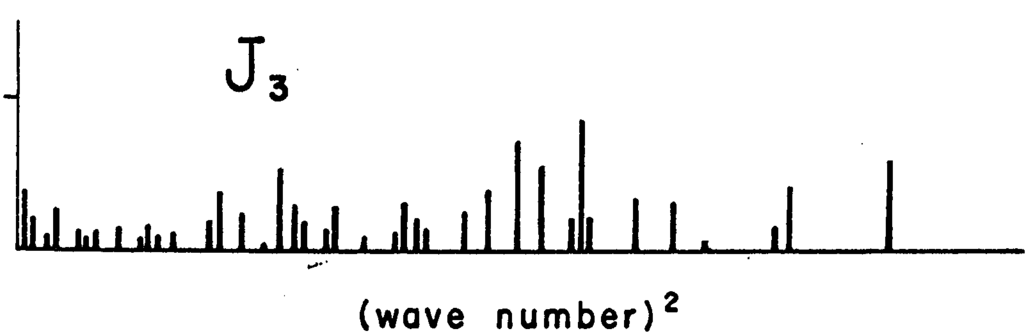


Fig. 9

3. The inertia term in the momentum equation for nondivergent flow.

Because we are not dealing with purely nondivergent motion, we cannot use the scheme discussed in the last section directly as it stands. Moreover, we are using the momentum equation and not the vorticity equation. But, the large-scale atmospheric motions are indeed quasi-nondivergent. A scheme which is inadequate for purely nondivergent motion is almost certainly also inadequate for quasi-nondivergent motion.

Our approach, here, is to first construct a suitable finite difference analog to the inertia term in the momentum equation for non divergent flow, and then to generalize it to allow for divergence.

Let us begin with the finite difference nondivergent vorticity equation for a square grid in which $\mathbb{J}_7$ of the last section is used. For vorticity, we use the form

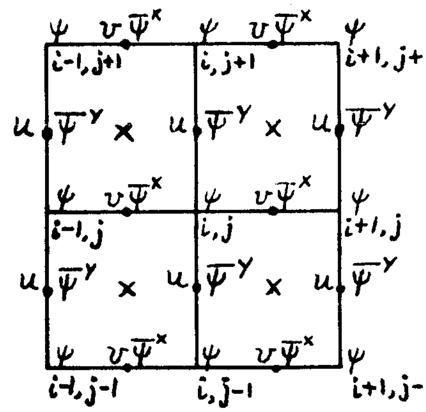
$$\zeta_{ij} = (\nabla^2\psi)_{ij} \equiv \frac{1}{d^2}(\psi_{i+1,j}+\psi_{i-1,j}+\psi_{i,j+1}+\psi_{i,j-1}-4\psi_{ij}) \ .$$

$$= \frac{1}{d}\left(\frac{\psi_{i+1,j}-\psi_{ij}}{d} - \frac{\psi_{ij}-\psi_{i-1,j}}{d} + \frac{\psi_{i,j+1}-\psi_{ij}}{d} - \frac{\psi_{ij}-\psi_{i,j-1}}{d}\right). \tag{V.1}$$

For the grid points shown in the accompanying figure, we define u, v, $\delta_y u$ and $\delta_x v$ by

$$u_{i,j+\frac{1}{2}} \equiv -\frac{\psi_{i,j+1}-\psi_{ij}}{d}, \qquad v_{i+\frac{1}{2},j} \equiv \frac{\psi_{i+1,j}-\psi_{ij}}{d}, \tag{V.2}$$

$$(\delta_y u)_{ij} \equiv u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}, \qquad (\delta_x v)_{ij} \equiv v_{i+\frac{1}{2},j} - v_{i-\frac{1}{2},j} \ . \tag{V.3}$$



Then the vorticity given by (V.1) is

$$\zeta_{ij} = \frac{1}{d}((\delta_x v)_{ij} - (\delta_y u)_{ij}) . \qquad \text{(V.1)'}$$

The vorticity equation may be written as

$$\frac{\partial}{\partial t}((\delta_x v)_{ij} - (\delta_y u)_{ij}) = \mathbb{J}_{ij}(\delta_x v - \delta_y u, \psi) . \qquad \text{(V.4)}$$

Here the symbol $\mathbb{J}$ is used for $\mathbb{J}_7$. Define $\bar{\psi}^x$ and $\bar{\psi}^y$ by

$$(\bar{\psi}^x)_{i+\frac{1}{2},j} = \tfrac{1}{2}(\psi_{ij} + \psi_{i+1,j}) , \qquad (\bar{\psi}^y)_{i,j+\frac{1}{2}} = \tfrac{1}{2}(\psi_{i,j+1} + \psi_{ij}) . \qquad \text{(V.5)}$$

Consider $\mathbb{J}_{i,j+\frac{1}{2}}(u, \bar{\psi}^y)$. From a property of the Jacobian, which is maintained by $\mathbb{J}_7$,

$$\mathbb{J}_{i,j+\frac{1}{2}}(u,\bar{\psi}^y) = \mathbb{J}_{i,j+\frac{1}{2}}(u,\bar{\psi}^y + \tfrac{1}{2}ud) , \qquad \text{(V.6)}$$

$$\mathbb{J}_{i,j-\frac{1}{2}}(u,\bar{\psi}^y) = \mathbb{J}_{i,j-\frac{1}{2}}(u,\bar{\psi}^y - \tfrac{1}{2}ud) . \qquad \text{(V.6)'}$$

Note that $(\bar{\psi}^y + \frac{1}{2}ud)_{i,j+\frac{1}{2}} = (\bar{\psi}^y - \frac{1}{2}ud)_{i,j-\frac{1}{2}} = \psi_{ij}$ for arbitrary i, j. Then, from (V.6) and (V.6)',

$$(\delta_y \mathbb{J}(u,\bar{\psi}^y))_{ij} \equiv \mathbb{J}_{i,j+\frac{1}{2}}(u,\bar{\psi}^y) - \mathbb{J}_{i,j-\frac{1}{2}}(u,\bar{\psi}^y)$$

$$= \mathbb{J}_{ij}(\delta_y u, \psi) . \qquad \text{(V.7)}$$

Similarly,

$$(\delta_x \mathbb{J}(v,\overline{\psi}^x))_{ij} = \mathbb{J}_{ij}(\delta_x v,\psi) \ . \qquad (V.7)'$$

(V.7) and (V.7)' are analogs, respectively, of

$$\frac{\partial}{\partial y} J(u,\psi) = J(\frac{\partial u}{\partial y}, \psi) \ ,$$

$$\frac{\partial}{\partial x} J(v,\psi) = J(\frac{\partial v}{\partial x}, \psi) \ .$$

From (V.7), (V.7)' and (V.1)',

$$(\delta_x \mathbb{J}(v,\overline{\psi}^x))_{ij} - (\delta_y \mathbb{J}(u,\overline{\psi}^y))_{ij} = \mathbb{J}_{ij}(\delta_x v - \delta_y u, \psi)$$

$$= d\mathbb{J}_{ij}(\zeta,\psi) \ . \qquad (V.8)$$

We conclude that

	$\mathbb{J}(u,\overline{\psi}^y)$	for $-\mathbb{V}\cdot\nabla u$ at the u-points
and	$\mathbb{J}(v,\overline{\psi}^x)$	for $-\mathbb{V}\cdot\nabla v$ at the v-points

are consistent with

$\mathbb{J}(\zeta,\psi)$ for $-\mathbb{V}\cdot\nabla\zeta$ at the ψ-points .

For purely nondivergent flow, the pressure must satisfy the balance equation, which is the divergence equation applied to nondivergent flow. The most logical place to carry pressure is then at the x points in the above figure, where the divergence is most simply defined. This configuration is also suitable when we are treating pure gravity waves, without (or with small) coriolis force. However, we already know that this configuration, which corresponds to scheme C in section 1, is not suitable for the geostrophic adjustment process by the dispersive gravity-inertia waves when the radius of deformation is small.

For the design of the general circulation model, we are using the configuration which corresponds to scheme B. We therefore sacrifice exact consistency with the finite difference vorticity equation. The reason that we do this is that if the geostrophic adjustment does not operate properly, the simulated flow will not be quasi-geostrophic, and, therefore, it will not necessarily be quasi-nondivergent. In that case, there would be no point in requiring an exact consistency with the nondivergent finite difference vorticity equation.

The integral constraint on mean square vorticity is effective in preventing a spurious energy cascade, because vorticity is a higher order derivative. Because of this, a similar constraint on the inertia term, not necessarily equivalent to the mean square vorticity conservation, should (and does) also prevent a spurious energy cascade.

For the differential case, we have

$$\overline{uJ(u, \psi) = 0} \; . \tag{V.9}$$

In addition, we have

$$\overline{\frac{\partial u}{\partial y} \frac{\partial}{\partial y} J(u, \psi)} = 0 \; , \tag{V.10}$$

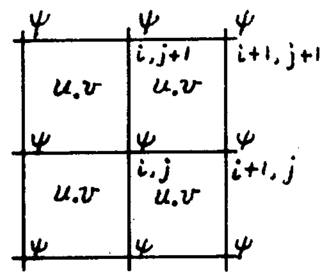
because

$$\frac{\partial}{\partial y} J(u, \psi) = J(\frac{\partial u}{\partial y} \, , \psi) \; . \tag{V.11}$$

Let ψ be carried by those points shown in the accompanying figure.

Define $\overline{\psi}$ by

$$(\overline{\psi})_{i+\frac{1}{2},j+\frac{1}{2}} \equiv \frac{1}{4}(\psi_{ij} + \psi_{i+1,j} + \psi_{i,j+1} + \psi_{i+1,j+1}) \ . \qquad \text{(V.12)}$$



Here the stream function ψ is defined by

$$\begin{aligned} u_{i+\frac{1}{2},j+\frac{1}{2}} &= -\frac{1}{2d}(\psi_{i+1,j+1} + \psi_{i,j+1} - \psi_{i+1,j} - \psi_{ij}) \\ v_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{2d}(\psi_{i+1,j+1} + \psi_{i+1,j} - \psi_{i,j+1} - \psi_{ij}) \ . \end{aligned} \qquad \text{(V.13)}$$

It can be shown that use of $\mathbb{J}_7$ for $\mathbb{J}_{i+\frac{1}{2},j+\frac{1}{2}}(u, \overline{\psi})$ maintains the constraints given by (V.9) and (V.10). Similarly, use of $\mathbb{J}_7$ for $\mathbb{J}_{i+\frac{1}{2},j+\frac{1}{2}}(v,\overline{\psi})$ maintains the constraints given by

$$\overline{v\, J(v,\psi)} = 0 \ , \qquad \text{(V.14)}$$

$$\overline{\frac{\partial v}{\partial y} \frac{\partial}{\partial y} J(v,\psi)} = 0 . \qquad \text{(V.15)}$$

The scheme for $-\mathbb{V}\cdot\nabla u$ and $\mathbb{V}\cdot\nabla v$, which will be given in the next chapter, reduces to this Jacobian when the mass flux is nondivergent.

VI. HORIZONTAL DIFFERENCING

1. The governing equations in orthogonal curvilinear coordinates.

Let the orthogonal curvilinear coordinates be ξ and η. The general circulation model uses the spherical coordinates, $\xi = \lambda$ (longitude) and $\eta = \varphi$ (latitude).

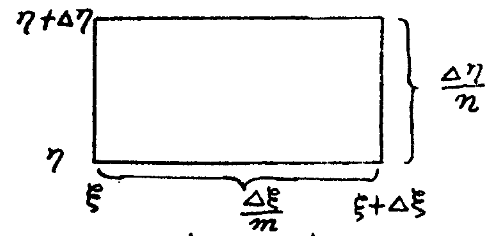
Let the actual distance corresponding to $d\xi$ be $(ds)_\xi$. Put

$$(ds)_\xi = \frac{1}{m} d\xi \; . \qquad \text{(VI.1)}$$

and

$$(ds)_\eta = \frac{1}{n} d\eta \; . \qquad \text{(VI.2)}$$

Consider a rectangular area element in the $\xi - \eta$ plane.



The actual lengths of the sides are $\frac{\Delta\xi}{m}$ and $\frac{\Delta\eta}{n}$. The area is $\frac{1}{mn}\Delta\xi\Delta\eta$.

Let the component of $\mathbb{V}$ in ξ be u and the component of $\mathbb{V}$ in η be v. The divergence is

$$\left[\delta_\xi\left(u\frac{\Delta\eta}{n}\right) + \delta_\eta\left(v\frac{\Delta\xi}{m}\right)\right] \Big/ \left(\frac{\Delta\xi}{m}\frac{\Delta\eta}{n}\right) , \qquad \text{(VI.3)}$$

where δ_ξ and δ_η are increments in ξ and η directions, respectively. In the limit as $\Delta\xi$, $\Delta\eta \rightarrow 0$, (VI.3) becomes

$$mn\left[\frac{\partial}{\partial\xi}\left(\frac{u}{n}\right) + \frac{\partial}{\partial\eta}\left(\frac{v}{m}\right)\right]. \qquad \text{(VI.3)'}$$

Similarly, the vorticity becomes

$$mn\left[\frac{\partial}{\partial\xi}\left(\frac{v}{n}\right) - \frac{\partial}{\partial\eta}\left(\frac{u}{m}\right)\right]. \qquad \text{(VI.4)}$$

The equation of continuity

In view of (VI.3)', the equation of continuity (I.16) may be written as

$$\frac{\partial}{\partial t}\left(\frac{\pi}{mn}\right) + \frac{\partial}{\partial \xi}\left(\pi \frac{u}{n}\right) + \frac{\partial}{\partial \eta}\left(\pi \frac{v}{m}\right) + \frac{\partial}{\partial \sigma}\left(\frac{\pi\dot{\sigma}}{mn}\right) = 0 . \qquad \text{(VI.5)}$$

The equation of motion

The equation of motion (I.22) may be written as

$$\frac{\partial \mathbb{V}}{\partial t} + \dot{\sigma}\frac{\partial \mathbb{V}}{\partial \sigma} + (f+\zeta)\,\mathbb{k}\times\mathbb{V} + \nabla\left(\tfrac{1}{2}\mathbb{V}^2 + \Phi\right) + \sigma\alpha\nabla\pi = \mathbb{F} , \qquad \text{(VI.6)}$$

where

$$\zeta \equiv \mathbb{k}\cdot\nabla\times\mathbb{V} = mn\left[\frac{\partial}{\partial \xi}\left(\frac{v}{n}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{m}\right)\right] . \qquad \text{(VI.7)}$$

The ξ-component of (VI.6) is

$$\frac{\partial u}{\partial t} + \dot{\sigma}\frac{\partial u}{\partial \sigma} - \left[f + mn\left(\frac{\partial}{\partial \xi}\left(\frac{v}{n}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{m}\right)\right)\right] v$$

$$+ m\frac{\partial}{\partial \xi}\left(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \Phi\right) + m\sigma\alpha\frac{\partial \pi}{\partial \xi} = F_{\xi} . \qquad \text{(VI.8)}$$

Rearranging the terms,

$$\frac{\partial u}{\partial t} + mu\frac{\partial u}{\partial \xi} + nv\frac{\partial u}{\partial \eta} + \dot{\sigma}\frac{\partial u}{\partial \sigma} - \left[f + mn\left(v\frac{\partial}{\partial \xi}\frac{1}{n} - u\frac{\partial}{\partial \eta}\frac{1}{m}\right)\right] v$$

$$+ m\left[\frac{\partial \Phi}{\partial \xi} + \sigma\alpha\frac{\partial \pi}{\partial \xi}\right] = F_{\xi} . \qquad \text{(VI.9)}$$

Similarly,

$$\frac{\partial v}{\partial t} + mu\frac{\partial v}{\partial \xi} + nv\frac{\partial v}{\partial \eta} + \dot{\sigma}\frac{\partial v}{\partial \sigma} + \left[f + mn\left(v\frac{\partial}{\partial \xi}\frac{1}{n} - u\frac{\partial}{\partial \eta}\frac{1}{m}\right)\right] u$$

$$+ n\left[\frac{\partial \Phi}{\partial \eta} + \sigma\alpha\frac{\partial \pi}{\partial \eta}\right] = F_{\eta} . \qquad \text{(VI.9)'}$$

Combining (VI.5) and (VI.9), we obtain the flux form for u,

$$\frac{\partial}{\partial t}\left(\frac{\pi}{mn}u\right)+\frac{\partial}{\partial\xi}\left(\frac{\pi u}{n}u\right)+\frac{\partial}{\partial\eta}\left(\frac{\pi v}{m}u\right)+\frac{\partial}{\partial\sigma}\left(\frac{\pi\dot{\sigma}}{mn}u\right)$$

$$-\left[\frac{f}{mn}+\left(v\frac{\partial}{\partial\xi}\frac{1}{n}-u\frac{\partial}{\partial\eta}\frac{1}{m}\right)\right]\pi v$$

$$+\frac{\pi}{n}\left[\frac{\partial\Phi}{\partial\xi}+\sigma\alpha\frac{\partial\pi}{\partial\xi}\right]=\frac{\pi}{mn}F_{\xi}\,. \qquad \text{(VI.10)}$$

Similarly,

$$\frac{\partial}{\partial t}\left(\frac{\pi}{mn}v\right)+\frac{\partial}{\partial\xi}\left(\frac{\pi u}{n}v\right)+\frac{\partial}{\partial\eta}\left(\frac{\pi v}{m}v\right)+\frac{\partial}{\partial\sigma}\left(\frac{\pi\dot{\sigma}}{mn}v\right)$$

$$+\left[\frac{f}{mn}+\left(v\frac{\partial}{\partial\xi}\frac{1}{n}-u\frac{\partial}{\partial\eta}\frac{1}{m}\right)\right]\pi u$$

$$+\frac{\pi}{m}\left[\frac{\partial\Phi}{\partial\eta}+\sigma\alpha\frac{\partial\pi}{\partial\eta}\right]=\frac{\pi}{mn}F_{\eta}\,. \qquad \text{(VI.11)}$$

From this point on, we will consider only those coordinate systems, such as the spherical and the cylindrical coordinate systems, in which m and n do not depend on ξ .

From (VI.10) we obtain the (relative) angular momentum equation

$$\frac{\partial}{\partial t}\left(\frac{\pi}{mn}\frac{u}{m}\right)+\frac{\partial}{\partial\xi}\left(\frac{\pi u}{n}\frac{u}{m}\right)+\frac{\partial}{\partial\eta}\left(\frac{\pi v}{m}\frac{u}{m}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\pi\dot{\sigma}}{mn}\frac{u}{m}\right)$$

$$-\left[\frac{f}{mn}\frac{\pi v}{m}\right]+\pi\left[\frac{\partial}{\partial\xi}\frac{\Phi}{mn}+\sigma\alpha\frac{\partial}{\partial\xi}\frac{\pi}{mn}\right]=\frac{\pi}{mn}\frac{F_{\xi}}{m}\,. \qquad \text{(VI.12)}$$

<u>The first law of thermodynamics</u>

(I.26) can be written as

$$\frac{\partial}{\partial t}\left(\frac{\pi}{mn}c_pT\right)+\frac{\partial}{\partial\xi}\left(\frac{\pi u}{n}c_pT\right)+\frac{\partial}{\partial\eta}\left(\frac{\pi v}{m}c_pT\right)+p^{\varkappa}\frac{\partial}{\partial\sigma}\left(\frac{\pi\dot{\sigma}}{mn}c_p\theta\right)$$

$$=\pi\sigma\alpha\left(\frac{\partial}{\partial t}\left(\frac{\pi}{mn}\right)+\frac{u}{n}\frac{\partial\pi}{\partial\xi}+\frac{v}{m}\frac{\partial\pi}{\partial\eta}\right)+\frac{\pi}{mn}Q\,. \qquad \text{(VI.13)}$$

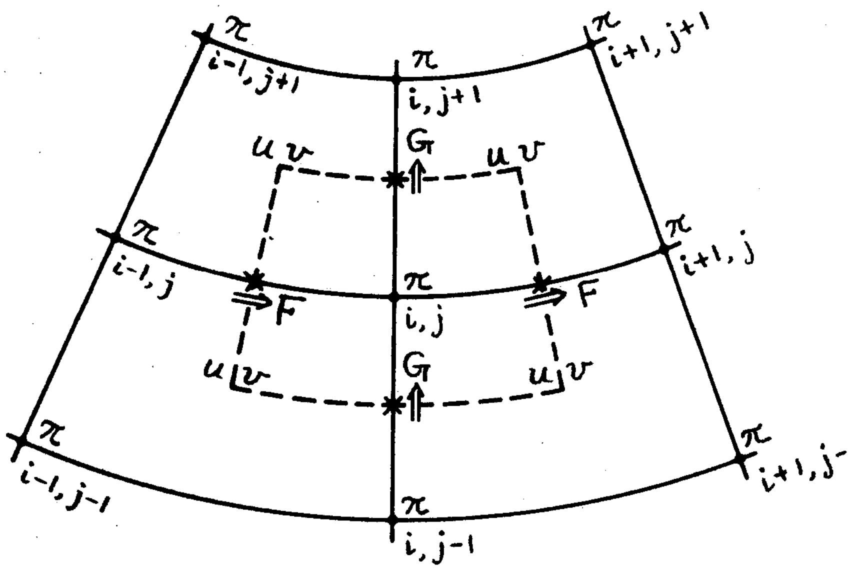
2. The equation of continuity.

We use the same distribution of the variables over the grid points as in scheme B of section I, chapter V.

The equation of continuity is

$$\frac{\partial}{\partial t}\left(\frac{\pi}{mn}\right) + \frac{\partial}{\partial \xi}\left(\pi \frac{u}{n}\right) + \frac{\partial}{\partial \eta}\left(\pi \frac{v}{m}\right) + \frac{\partial}{\partial \sigma}\left(\frac{\pi\dot{\sigma}}{mn}\right) = 0 \ . \tag{VI.5}$$

For the spherical grid, $\xi = \lambda$, $\eta = \varphi$, $\frac{1}{m} = a \cos\varphi$ and $\frac{1}{n} = a$.



We use the following form:

$$\frac{\partial \Pi_{ij}}{\partial t} + F^{k}_{i+\frac{1}{2},j} - F^{k}_{i-\frac{1}{2},j} + G^{k}_{i,j+\frac{1}{2}} - G^{k}_{i,j-\frac{1}{2}} + \frac{1}{\Delta\sigma_k}\left(\dot{S}^{k+1}_{ij} - \dot{S}^{k-1}_{ij}\right) = 0 \ , \tag{VI.14}$$

where

$$\Pi \equiv \pi \frac{\Delta\xi\Delta\eta}{mn} , \qquad F \equiv \pi u \frac{\Delta\eta}{n} , \qquad G \equiv \pi v \frac{\Delta\xi}{m} , \qquad \dot{S} \equiv \Pi\dot{\sigma} \ , \tag{VI.15}$$

and the vertical index k now appears as a superscript.

For the mass fluxes F and G, the following forms are used:

$$F^k_{i+\frac{1}{2},j} = \frac{1}{4}\overline{\left[\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} + \left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j-\frac{1}{2}}\right](\pi_{i+1,j} + \pi_{ij})}\ , \tag{VI.16}$$

where

$$\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} \equiv u^k_{i+\frac{1}{2},j+\frac{1}{2}}\left(\frac{\Delta\eta}{n}\right)_{j+\frac{1}{2}}\ . \tag{VI.17}$$

For the time being, ignore the superior bar operator, which is a linear smoothing operator in ξ. The form and the role of this operator will be described in Chapter VII, section 2.

Similarly,

$$G^k_{i,j+\frac{1}{2}} = \frac{1}{4}\left[\left(v\frac{\Delta\xi}{m}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} + \left(v\frac{\Delta\xi}{m}\right)^k_{i-\frac{1}{2},j+\frac{1}{2}}\right](\pi_{i,j+1} + \pi_{ij})\ , \tag{VI.18}$$

where

$$\left(v\frac{\Delta\xi}{m}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} \equiv v^k_{i+\frac{1}{2},j+\frac{1}{2}}\left(\frac{\Delta\xi}{m}\right)_{j+\frac{1}{2}} \tag{VI.19}$$

3. <u>The pressure gradient force</u>.

As in (VI.10), the pressure gradient force in the ξ-direction is

$$-\frac{\pi}{n}\left[\frac{\partial\Phi}{\partial\xi} + \sigma\alpha\frac{\partial\pi}{\partial\xi}\right]\ . \tag{VI.22}$$

For the first term, we choose the form:

$$-\left(\frac{\pi}{n}\frac{\partial\Phi}{\partial\xi}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} = -\frac{1}{\Delta\xi\Delta\eta}\frac{\Delta\eta}{n_{j+\frac{1}{2}}}\frac{1}{4}\left[\overline{(\pi_{i+1,j+1} + \pi_{i,j+1})(\Phi^k_{i+1,j+1} - \Phi^k_{i,j+1})} + \overline{(\pi_{i+1,j} + \pi_{i,j})(\Phi^k_{i+1,j} - \Phi^k_{i,j})}\right]. \tag{VI.23}$$

Continue, for the time being, to ignore the bar operator.

Corresponding to the relation $-\frac{\pi}{n}\frac{\partial\Phi}{\partial\xi} = \frac{1}{n}\left[\frac{\partial}{\partial\xi}(\pi\Phi) - \Phi\frac{\partial\pi}{\partial\xi}\right]$, we can rewrite (VI.23) as follows:

$$-\left(\frac{\pi}{n}\frac{\partial\Phi}{\partial\xi}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} = -\frac{1}{\Delta\xi\Delta\eta}\frac{\Delta\eta}{n_{j+\frac{1}{2}}}\frac{1}{2}$$

$$\times\left[\overline{\pi_{i+1,j+1}\Phi^k_{i+1,j+1} - \pi_{i,j+1}\Phi^k_{i,j+1} - \tfrac{1}{2}(\Phi^k_{i+1,j+1} + \Phi^k_{i,j+1})(\pi_{i+1,j+1} - \pi_{i,j+1})}\right.$$

$$\left.+\overline{\pi_{i+1,j}\Phi^k_{i+1,j} - \pi_{ij}\Phi^k_{ij} - \tfrac{1}{2}(\Phi^k_{i+1,j} + \Phi^k_{ij})(\pi_{i+1,j} - \pi_{ij})}\right]. \qquad \text{(VI.24)}$$

To be consistent with (VI.24), we choose the following form for the second term in (VI.22).

$$-\left(\frac{\pi}{n}\sigma\alpha\frac{\partial\pi}{\partial\xi}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} = -\frac{1}{\Delta\xi\Delta\eta}\frac{\Delta\eta}{n_{j+\frac{1}{2}}}\frac{1}{4}$$

$$\times\left[\overline{\left((\pi\sigma\alpha)^k_{i+1,j+1} + (\pi\sigma\alpha)^k_{i,j+1}\right)(\pi_{i+1,j+1} - \pi_{i,j+1})}\right.$$

$$\left.+\overline{\left((\pi\sigma\alpha)^k_{i+1,j} + (\pi\sigma\alpha)^k_{ij}\right)(\pi_{i+1,j} - \pi_{ij})}\right], \qquad \text{(VI.25)}$$

where

$$(\pi\sigma\alpha)^k_{ij} = \pi_{ij}\,\sigma^k\,\frac{RT^k_{ij}}{p^k_{ij}}. \qquad \text{(VI.26)}$$

At each grid point we apply (III.16) to $\pi\sigma\alpha$ in (VI.25). Adding (VI.25) to (VI.24), we obtain a form which corresponds to $-\frac{1}{n}\left(\frac{\partial(\pi\Phi)}{\partial\xi} - \frac{\partial}{\partial\sigma}(\Phi\sigma)\frac{\partial\pi}{\partial\xi}\right)$, from which we can readily show that the integral properties, discussed in section 2, chapter II and section 5, chapter III, are maintained. The momentum generation at the grid point $i+\frac{1}{2}$, $j+\frac{1}{2}$ by the slope of the earth's surface in ξ-direction is given by

$$-\frac{1}{g}\frac{\Delta\eta}{n_{j+\frac{1}{2}}}\frac{1}{4}\Big[(\pi_{i+1,j+1}+\pi_{i,j+1})(\Phi^s_{i+1,j+1}-\Phi^s_{i,j+1})$$

$$+(\pi_{i+1,j}+\pi_{i,j})(\Phi^s_{i+1,j}-\Phi^s_{i,j})\Big]\ , \qquad \text{(VI.27)}$$

where $\Phi^s_{ij}=(gz_s)_{ij}$.

In summary, the pressure gradient force which contributes to $\frac{\partial}{\partial t}(\Pi u)^k_{i+\frac{1}{2},j+\frac{1}{2}}$ is

$$-\frac{\Delta\eta}{n_{j+\frac{1}{2}}}\frac{1}{4}\Big[\overline{(\pi_{i+1,j+1}+\pi_{i,j+1})(\Phi^k_{i+1,j+1}-\Phi^k_{i,j+1})}$$

$$+\overline{(\pi_{i+1,j}+\pi_{i,j})(\Phi^k_{i+1,j}-\Phi^k_{i,j})}$$

$$+\overline{\left((\pi\sigma\alpha)^k_{i+1,j+1}+(\pi\sigma\alpha)^k_{i,j+1}\right)(\pi_{i+1,j+1}-\pi_{i,j+1})}$$

$$+\overline{\left((\pi\sigma\alpha)^k_{i+1,j}+(\pi\sigma\alpha)^k_{i,j}\right)(\pi_{i+1,j}-\pi_{i,j})}\Big]\ . \qquad \text{(VI.28)}$$

Similarly, the pressure gradient force which contributes to $\frac{\partial}{\partial t}(\Pi v)^k_{i+\frac{1}{2},j+\frac{1}{2}}$ is

$$-\frac{\Delta\xi}{m_{j+\frac{1}{2}}}\frac{1}{4}\Big[(\pi_{i+1,j+1}+\pi_{i+1,j})(\Phi^k_{i+1,j+1}-\Phi^k_{i+1,j})$$

$$+(\pi_{i,j+1}+\pi_{i,j})(\Phi^k_{i,j+1}-\Phi^k_{i,j})$$

$$+\left((\pi\sigma\alpha)^k_{i+1,j+1}+(\pi\sigma\alpha)^k_{i+1,j}\right)(\pi_{i+1,j+1}-\pi_{i+1,j}) \qquad \text{(VI.29)}$$

$$+\left((\pi\sigma\alpha)^k_{i,j+1}+(\pi\sigma\alpha)^k_{i,j}\right)\left(\pi_{i,j+1}-\pi_{i,j}\right)\Big]\ .$$

From (VI.26), $\qquad (\pi\sigma\alpha)^k_{ij}=\pi_{ij}\,\sigma^k\,\dfrac{RT^k_{ij}}{p^k_{ij}}$.

$\Pi\equiv\pi\,\dfrac{\Delta\xi\,\Delta\eta}{mn}$. However, Π at the u,v-points has not yet been defined.

4. Kinetic energy generation and the first law of thermodynamics.

The contribution of the pressure gradient force to the kinetic energy generation, $\frac{\partial}{\partial t}(\Pi \frac{1}{2} u^2)^k_{i+\frac{1}{2},j+\frac{1}{2}}$, is obtained by multiplying (VI.28) by $u^k_{i+\frac{1}{2},j+\frac{1}{2}}$. The contribution of the gradients of Φ and π at j to the kinetic energy generation is

$$-\frac{1}{4}\left(u\frac{\Delta\eta}{n}\right)_{i+\frac{1}{2},j+\frac{1}{2}} \times \overline{\left[(\pi_{i+1,j}+\pi_{i,j})(\Phi^k_{i+1,j}-\Phi^k_{i,j})+\left((\pi\sigma\alpha)^k_{i+1,j}+(\pi\sigma\alpha)^k_{i,j}\right)(\pi_{i+1,j}-\pi_{i,j})\right]} \; . \tag{VI.30}$$

However, the gradients of Φ and π inside the brackets of (VI.30) also contribute to the kinetic energy generation $\frac{\partial}{\partial t}(\Pi \frac{1}{2} u^2)^k_{i+\frac{1}{2},j-\frac{1}{2}}$. The combined effect is

$$-\frac{1}{4}\left(\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}}+\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j-\frac{1}{2}}\right) \times \overline{\left[(\pi_{i+1,j}+\pi_{i,j})(\Phi^k_{i+1,j}-\Phi^k_{i,j})+\left((\pi\sigma\alpha)^k_{i+1,j}+(\pi\sigma\alpha)^k_{i,j}\right)(\pi_{i+1,j}-\pi_{i,j})\right]} \; . \tag{VI.31}$$

As already indicated, the superior bar denotes a linear smoothing operator in ξ. From the form of the operator, we can show that the difference of (VI.31) from

$$-\frac{1}{4}\overline{\left(\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}}+\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j-\frac{1}{2}}\right)} \times\left[(\pi_{i+1,j}+\pi_{ij})(\Phi^k_{i+1,j}-\Phi^k_{ij})+\left((\pi\sigma\alpha)^k_{i+1,j}+(\pi\sigma\alpha)^k_{ij}\right)(\pi_{i+1,j}-\pi_{i,j})\right] \tag{VI.32}$$

vanishes when the summation over all i is taken. In other words,

$$\sum_i R_{i+\frac{1}{2}} = 0 \; , \tag{VI.33}$$

where

$$R_{i+\frac{1}{2}} \equiv \text{(VI.31)} - \text{(VI.32)} \ , \qquad \text{(VI.33)'}$$

and $\sum_i$ is the summation over all i. Using the definition of F given by (VI.16), we obtain

$$\text{(VI.32)} = - F^k_{i+\frac{1}{2},j}\left(\Phi^k_{i+1,j} - \Phi^k_{ij}\right) - \frac{1}{4}\left(\overline{\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} + \left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j-\frac{1}{2}}}\right)\left((\pi\sigma\alpha)^k_{i+1,j} + (\pi\sigma\alpha)^k_{ij}\right)\left(\pi_{i+1,j} - \pi_{ij}\right)\Big) . \qquad \text{(VI.34)}$$

Further, we can show that

$$\sum_i \text{(VI.34)} = \sum_i \Big[\left(F^k_{i+\frac{1}{2},j} - F^k_{i-\frac{1}{2},j}\right)\Phi^k_{ij}$$
$$- \frac{1}{8}\left(\overline{\left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} + \left(u\frac{\Delta\eta}{n}\right)^k_{i+\frac{1}{2},j-\frac{1}{2}}}\right)\left((\pi\sigma\alpha)^k_{i+1,j} + (\pi\sigma\alpha)^k_{ij}\right)\left(\pi_{i+1,j} - \pi_{ij}\right)$$
$$- \frac{1}{8}\left(\overline{\left(u\frac{\Delta\eta}{n}\right)^k_{i-\frac{1}{2},j+\frac{1}{2}} + \left(u\frac{\Delta\eta}{n}\right)^k_{i-\frac{1}{2},j-\frac{1}{2}}}\right)\left((\pi\sigma\alpha)^k_{ij} + (\pi\sigma\alpha)^k_{i-1,j}\right)\left(\pi_{ij} - \pi_{i-1,j}\right)\Big] . \qquad \text{(VI.35)}$$

Similarly, the contribution of the gradients of Φ and π at i to $\frac{\partial}{\partial t}(\Pi \frac{1}{2} v^2)$ is given by

$$\sum_j \Big[\left(G^k_{i,j+\frac{1}{2}} - G^k_{i,j-\frac{1}{2}}\right)\Phi^k_{ij}$$
$$- \frac{1}{8}\left(\left(v\frac{\Delta\xi}{m}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} + \left(v\frac{\Delta\xi}{m}\right)^k_{i-\frac{1}{2},j+\frac{1}{2}}\right)\left((\pi\sigma\alpha)^k_{i,j+1} + (\pi\sigma\alpha)^k_{ij}\right)\left(\pi_{i,j+1} - \pi_{ij}\right)$$
$$- \frac{1}{8}\left(\left(v\frac{\Delta\xi}{m}\right)^k_{i+\frac{1}{2},j-\frac{1}{2}} + \left(v\frac{\Delta\xi}{m}\right)^k_{i-\frac{1}{2},j-\frac{1}{2}}\right)\left((\pi\sigma\alpha)^k_{ij} + (\pi\sigma\alpha)^k_{i,j-1}\right)\left(\pi_{ij} - \pi_{i,j-1}\right)\Big] . \qquad \text{(VI.35)'}$$

$\sum_j$ (VI.35) + $\sum_i$ (VI.35)', with the equation of continuity (VI.14),

gives the contribution of Φ_{ij}^{k} to the kinetic energy generation,

$$-\left[\frac{\partial \Pi_{ij}}{\partial t} + \frac{1}{\Delta\sigma_k}(\dot{S}_{ij}^{k+1} - \dot{S}_{ij}^{k-1})\right]\Phi_{ij}^{k} \tag{VI.36}$$

Further, following the process which led to (III.18), we obtain the finite difference expression for $\omega\alpha$. Using that expression, the thermodynamic energy equation (III.40) may be written as

$$\frac{\partial}{\partial t}(\Pi_{ij} T_{ij}^{k}) + F_{i+\frac{1}{2},j}^{k}\frac{T_{i+1,j}^{k}+T_{i,j}^{k}}{2} - F_{i-\frac{1}{2},j}^{k}\frac{T_{i,j}^{k}+T_{i-1,j}^{k}}{2}$$

$$+ G_{i,j+\frac{1}{2}}^{k}\frac{T_{i,j+1}^{k}+T_{i,j}^{k}}{2} - G_{i,j-\frac{1}{2}}^{k}\frac{T_{i,j}^{k}+T_{i,j-1}^{k}}{2}$$

$$+ \frac{1}{\Delta\sigma_k}\left[\dot{S}_{ij}^{k+1}(p_{ij}^{k})^{\varkappa}\theta_{ij}^{k+1} - \dot{S}_{ij}^{k-1}(p_{ij}^{k})^{\varkappa}\theta_{ij}^{k-1}\right]$$

$$= \frac{1}{c_p}\left[(\pi\sigma\alpha)_{ij}^{k}\frac{\partial \Pi_{ij}}{\partial t}\right.$$

$$+\frac{1}{8}\left\{\overline{\left((u\frac{\Delta\eta}{n})_{i+\frac{1}{2},j+\frac{1}{2}}^{k} + (u\frac{\Delta\eta}{n})_{i+\frac{1}{2},j-\frac{1}{2}}^{k}\right)}\left((\pi\sigma\alpha)_{i+1,j}^{k} + (\pi\sigma\alpha)_{ij}^{k}\right)(\pi_{i+1,j}-\pi_{ij})\right.$$

$$+ \overline{\left((u\frac{\Delta\eta}{n})_{i-\frac{1}{2},j+\frac{1}{2}}^{k} + (u\frac{\Delta\eta}{n})_{i-\frac{1}{2},j-\frac{1}{2}}^{k}\right)}\left((\pi\sigma\alpha)_{ij}^{k} + (\pi\sigma\alpha)_{i-1,j}^{k}\right)(\pi_{ij}-\pi_{i-1,j})$$

$$+ \left((v\frac{\Delta\xi}{m})_{i+\frac{1}{2},j+\frac{1}{2}}^{k} + (v\frac{\Delta\xi}{m})_{i-\frac{1}{2},j+\frac{1}{2}}^{k}\right)\left((\pi\sigma\alpha)_{i,j+1}^{k} + (\pi\sigma\alpha)_{ij}^{k}\right)(\pi_{i,j+1}-\pi_{ij})$$

$$+ \left.\left((v\frac{\Delta\xi}{m})_{i+\frac{1}{2},j-\frac{1}{2}}^{k} + (v\frac{\Delta\xi}{m})_{i-\frac{1}{2},j-\frac{1}{2}}^{k}\right)\left((\pi\sigma\alpha)_{ij}^{k} + (\pi\sigma\alpha)_{i,j-1}^{k}\right)(\pi_{ij} - \pi_{i,j-1})\right\}$$

$$\left. + \Pi_{ij} Q_{ij}\right]. \tag{VI.37}$$

5. The water vapor equation.

The difficulty pointed out in sec. 4, Chap. IV exists even with horizontal differencing. However, it is much less serious in the horizontal differencing so that we might simply use

$$\frac{\partial}{\partial t}(\Pi_{ij}q^{k}_{ij}) + F^{k}_{i+\frac{1}{2},j}\,\frac{q^{k}_{i+1,j}+q^{k}_{ij}}{2} - F_{i-\frac{1}{2},j}\,\frac{q^{k}_{ij}+q^{k}_{i-1,j}}{2}$$

$$+ G^{k}_{i,j+\frac{1}{2}}\,\frac{q^{k}_{i,j+1}+q^{k}_{ij}}{2} - G^{k}_{i,j-\frac{1}{2}}\,\frac{q^{k}_{ij}+q^{k}_{i,j-1}}{2}$$

$$+ \frac{1}{\Delta\sigma_k}\left[\dot{S}^{k+1}_{ij}\,q^{k+1}_{ij} - \dot{S}^{k-1}_{ij}\,q^{k-1}_{ij}\right]$$

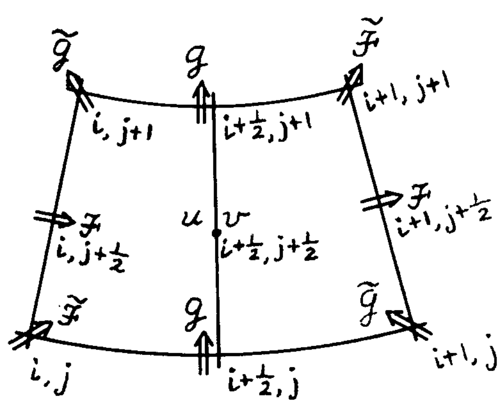
$$= \Pi_{ij}(-C + E)\,. \qquad \text{(VI.38)}$$

But the arithmetic means should be replaced by zero when there is mass flux out of the grid point where q is already zero. Possibilities other than (VI.38) are being tested (August 1971).

6. Momentum fluxes.

The form we chose for $\frac{\partial}{\partial t}(\pi\,\frac{\Delta\xi\,\Delta\eta}{mn}\,u) + \Delta\xi\,\frac{\partial}{\partial\xi}(\pi u\,\frac{\Delta\eta}{n}u) + \Delta\eta\,\frac{\partial}{\partial\eta}(\pi v\,\frac{\Delta\xi}{m}u)$ $+ \frac{\partial}{\partial\sigma}(\pi\dot{\sigma}\,\frac{\Delta\xi\Delta\eta}{mn}u)$ is

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\Pi^{(u)}_{i+\frac{1}{2},j+\frac{1}{2}}\, u^{k}_{i+\frac{1}{2},j+\frac{1}{2}}\right) + \tfrac{1}{2}\Big[\frac{2}{3}\Big\{\mathcal{F}^{k}_{i+1,j+\frac{1}{2}}\left(u^{k}_{i+\frac{3}{2},j+\frac{1}{2}} + u^{k}_{i+\frac{1}{2},j+\frac{1}{2}}\right) \\
&\qquad\qquad - \mathcal{F}^{k}_{i,j+\frac{1}{2}}\left(u^{k}_{i+\frac{1}{2},j+\frac{1}{2}} + u^{k}_{i-\frac{1}{2},j+\frac{1}{2}}\right) \\
&\qquad\qquad + g^{k}_{i+\frac{1}{2},j+1}\left(u^{k}_{i+\frac{1}{2},j+\frac{3}{2}} + u^{k}_{i+\frac{1}{2},j+\frac{1}{2}}\right) \\
&\qquad\qquad - g^{k}_{i+\frac{1}{2},j}\left(u^{k}_{i+\frac{1}{2},j+\frac{1}{2}} + u^{k}_{i+\frac{1}{2},j-\frac{1}{2}}\right)\Big\} \\
&\quad + \tfrac{1}{3}\Big\{\tilde{\mathcal{F}}^{k}_{i+1,j+1}\left(u^{k}_{i+\frac{3}{2},j+\frac{3}{2}} + u^{k}_{i+\frac{1}{2},j+\frac{1}{2}}\right) - \tilde{\mathcal{F}}^{k}_{i,j}\left(u^{k}_{i+\frac{1}{2},j+\frac{1}{2}} + u^{k}_{i-\frac{1}{2},j-\frac{1}{2}}\right) \\
&\qquad + \tilde{g}^{k}_{i,j+1}\left(u^{k}_{i-\frac{1}{2},j+\frac{3}{2}} + u^{k}_{i+\frac{1}{2},j+\frac{1}{2}}\right) - \tilde{g}^{k}_{i+1,j}\left(u_{i+\frac{1}{2},j+\frac{1}{2}} + u_{i+\frac{3}{2},j-\frac{1}{2}}\right)\Big\}\Big] \\
&\quad + \frac{1}{\Delta\sigma_k}\,\tfrac{1}{2}\Big[\dot{S}^{(u)\,k+1}_{i+\frac{1}{2},j+\frac{1}{2}}\left(u^{k+2}_{i+\frac{1}{2},j+\frac{1}{2}} + u^{k}_{i+\frac{1}{2},j+\frac{1}{2}}\right) - \dot{S}^{(u)\,k-1}_{i+\frac{1}{2},j+\frac{1}{2}}\left(u^{k}_{i+\frac{1}{2},j+\frac{1}{2}} + u^{k-2}_{i+\frac{1}{2},j+\frac{1}{2}}\right)\Big],
\end{aligned}
\tag{VI.39}
$$



where

$$
\begin{aligned}
\mathcal{F}_{i+1,j+\frac{1}{2}} &= \tfrac{1}{4}\left(F_{i+\frac{3}{2},j+1} + F_{i+\frac{3}{2},j} + F_{i+\frac{1}{2},j+1} + F_{i+\frac{1}{2},j}\right) \\
g_{i+\frac{1}{2},j+1} &= \tfrac{1}{4}\left(G_{i+1,j+\frac{3}{2}} + G_{i+1,j+\frac{1}{2}} + G_{i,j+\frac{3}{2}} + G_{i,j+\frac{1}{2}}\right) \\
\tilde{\mathcal{F}}_{i+1,j+1} &= \tfrac{1}{4}\left(F_{i+\frac{3}{2},j+1} + F_{i+\frac{1}{2},j+1} + G_{i+1,j+\frac{3}{2}} + G_{i+1,j+\frac{1}{2}}\right) \\
\tilde{g}_{i,j+1} &= \tfrac{1}{4}\left(-F_{i-\frac{1}{2},j+1} - F_{i+\frac{1}{2},j+1} + G_{i,j+\frac{1}{2}} + G_{i,j+\frac{3}{2}}\right)
\end{aligned}
\tag{VI.40}
$$

We have not defined $\Pi^{(u)}$ yet. If we put u = const. both in space and time, (VI.39) must become zero. Then we get

$$\frac{\partial}{\partial t}\Pi^{(u)}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{2}{3}(\mathfrak{J}^{k}_{i+1,j+\frac{1}{2}}-\mathfrak{J}^{k}_{i,j+\frac{1}{2}}+g^{k}_{i+\frac{1}{2},j+1}-g^{k}_{i+\frac{1}{2},j})$$

$$+\tfrac{1}{3}(\widetilde{\mathfrak{J}}^{k}_{i+1,j+1}-\widetilde{\mathfrak{J}}^{k}_{ij}+\widetilde{g}^{k}_{i,j+1}-\widetilde{g}^{k}_{i+1,j}) \;.$$

$$+\frac{1}{\Delta\sigma_k}(\dot{S}^{(u)k+1}_{i+\frac{1}{2},j+\frac{1}{2}}-\dot{S}^{(u)k-1}_{i+\frac{1}{2},j+\frac{1}{2}}) = 0 \;. \tag{VI.41}$$

We derived (VI.41) rather formally. We can show that (VI.41) is necessary for maintaining the conservation of kinetic energy under a pure advective process.

From the definitions of $\mathfrak{J}$, g, $\widetilde{\mathfrak{J}}$ and $\widetilde{g}$ given by (VI.40), we can show that (VI.41) automatically holds if we define

$$\Pi^{(u)}_{i+\frac{1}{2},j+\frac{1}{2}} = \tfrac{1}{4}(\Pi_{i+1,j+1}+\Pi_{i+1,j}+\Pi_{i,j+1}+\Pi_{i,j}) \;, \tag{VI.42}$$

$$\dot{S}^{(u)}_{i+\frac{1}{2},j+\frac{1}{2}} = \tfrac{1}{4}(\dot{S}_{i+1,j}+\dot{S}_{i+1,j}+\dot{S}_{i,j+1}+\dot{S}_{ij}) \tag{VI.43}$$

For v, we can use the exactly same form.

7. Coriolis force.

See (VI.10). Coriolis force, plus the metric term which contributes to $\frac{\partial}{\partial t}(\Pi u)$ is

$$\left[f\,\frac{\Delta\xi\Delta\eta}{mn} - u\Delta\xi\,\Delta\eta\,\frac{\partial}{\partial\eta}\,\frac{1}{m}\right]\pi v \tag{VI.44}$$

and coriolis force which contributes to $\frac{\partial}{\partial t}(\Pi v)$ is

$$-\left[f \frac{\Delta\xi \, \Delta\eta}{mn} - u \, \Delta\xi \, \Delta\eta \frac{\partial}{\partial\eta} \frac{1}{m} \right] \pi u \qquad \text{(VI.45)}$$

Of course, kinetic energy is not generated by the coriolis force. We must maintain the relation (VI.44) x u + (VI.45) x v = 0. We use the following form for (VI.44) at the point $(i+\frac{1}{2}, j+\frac{1}{2})$.

$$\frac{1}{8}\Big[(\pi_{i+1,j+1} + \pi_{i+1,j})(C^k_{i+1,j+1} + C^k_{i+1,j})$$

$$+ (\pi_{i,j+1} + \pi_{i,j})(C^k_{i,j+1} + C^k_{i,j})\Big] v^k_{i+\frac{1}{2},j+\frac{1}{2}} \qquad \text{(VI.46)}$$

and similar form for (VI.45). Here

$$C^k_{ij} \equiv f_j \left(\frac{\Delta\xi\Delta\eta}{mn}\right)_j - \tfrac{1}{4}\left(u^k_{i+\frac{1}{2},j+\frac{1}{2}} + u^k_{i+\frac{1}{2},j-\frac{1}{2}} + u^k_{i-\frac{1}{2},j+\frac{1}{2}} + u^k_{i-\frac{1}{2},j-\frac{1}{2}}\right)$$

$$\times \left(\left(\frac{\Delta\xi}{m}\right)_{j+\frac{1}{2}} - \left(\frac{\Delta\xi}{m}\right)_{j-\frac{1}{2}} \right) \qquad \text{(VI.47)}$$

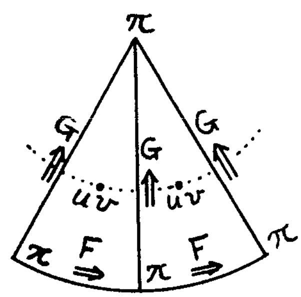
When C is constant in i, that is, when the metric term is either negligible or constant in i, the zonal average of (VI.46) is equal to the zonal average of the mass flux G, except a constant factor. This is desirable for avoiding spurious conversion from Ω-momentum to u-momentum.

VII. MODIFICATION OF THE HORIZONTAL DIFFERENCING NEAR THE POLES

1. Modification of the equations.

The poles are singular points of the spherical coordinates and the velocity components cannot be defined there. Therefore, we let the poles be π-points.

π at the poles must change as a result of the meridional mass flux, G, at all of the points on the latitude circle where the velocity components are carried, as shown by the dotted line in the figure.



To simplify the computation, we treat each pole as if it were a group of points. Each point has index i. At each i we apply the equation of continuity (VI.14). After computing $\frac{\partial \pi}{\partial t}$ and $\dot{S}$, at all of the points, we take the average.

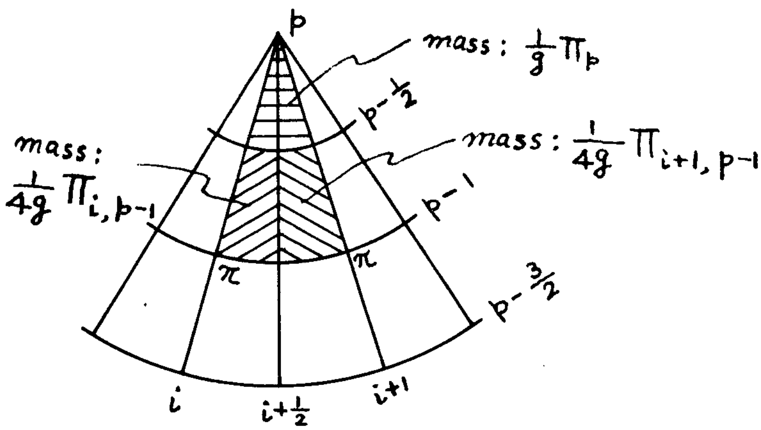
We apply the same procedure to the first law of thermodynamics. It follows that those terms in (VI.37) which are not defined at the poles make no contribution.

In the equation of motion, the momentum flux and coriolis force terms, but not the pressure gradient term, must be modified for the points next to the poles.

Let the north pole be $j = p$. We begin by modifying (VI.42). From geometrical considerations, we let

$$\Pi^{(u)}_{i+\frac{1}{2},p-\frac{1}{2}} = \Pi_p + \tfrac{1}{4}(\Pi_{i+1,p-1} + \Pi_{i,p-1}) \; , \qquad \text{(VII.1)}$$

$$\dot{S}^{(u)}_{i+\frac{1}{2},p-\frac{1}{2}} = \dot{S}_p + \tfrac{1}{4}(\dot{S}_{i+1,p-1} + \dot{S}_{i,p-1}) \; . \qquad \text{(VII.2)}$$



$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\Pi^{(u)}_{i+\frac{1}{2},p-\frac{1}{2}} u^k_{i+\frac{1}{2},p-\frac{1}{2}}\right) + \frac{1}{2}\Bigg[\frac{2}{3}\Bigg\{\mathcal{F}^{*k}_{i+1,p-\frac{1}{2}}\left(u^k_{i+\frac{3}{2},p-\frac{1}{2}} + u^k_{i+\frac{1}{2},p-\frac{1}{2}}\right) \\
&\qquad - \mathcal{F}^{*k}_{i,p-\frac{1}{2}}\left(u^k_{i+\frac{1}{2},p-\frac{1}{2}} + u^k_{i-\frac{1}{2},p-\frac{1}{2}}\right) \\
&\qquad - g^k_{i+\frac{1}{2},p-1}\left(u^k_{i+\frac{1}{2},p-\frac{1}{2}} + u_{i+\frac{1}{2},p-\frac{3}{2}}\right)\Bigg\} \\
&+ \frac{1}{3}\left\{-\tilde{\mathcal{F}}^k_{i,p-1}\left(u_{i+\frac{1}{2},p-\frac{1}{2}} + u_{i-\frac{1}{2},p-\frac{3}{2}}\right) - \tilde{g}^k_{i+1,p-1}\left(u_{i+\frac{1}{2},p-\frac{1}{2}} + u_{i+\frac{3}{2},p-\frac{3}{2}}\right)\right\}\Bigg] \\
&+ \frac{1}{\Delta\sigma_k}\frac{1}{2}\left[\dot{S}^{(u)k+1}_{i+\frac{1}{2},p-\frac{1}{2}}\left(u^{k+2}_{i+\frac{1}{2},p-\frac{1}{2}} + u^k_{i+\frac{1}{2},p-\frac{1}{2}}\right) - \dot{S}^{(u)k-1}_{i+\frac{1}{2},p-\frac{1}{2}}\left(u^k_{i+\frac{1}{2},p-\frac{1}{2}} + u^{k-2}_{i+\frac{1}{2},p-\frac{1}{2}}\right)\right] .
\end{aligned}$$

(VII.3)

For energy conservation, we require

$$\frac{\partial \mathfrak{M}^{(u)}_{i+\frac{1}{2},p-\frac{1}{2}}}{\partial t} + \Big[\frac{2}{3} \left(\mathfrak{F}^{*k}_{i+1,p-\frac{1}{2}} - \mathfrak{F}^{*k}_{i,p-\frac{1}{2}} - g^{k}_{i+\frac{1}{2},p-1} \right)$$

$$- \tfrac{1}{3} \left(\tilde{\mathfrak{F}}^{k}_{i,p-1} + \tilde{g}^{k}_{i+1,p-1} \right) \Big]$$

$$+ \frac{1}{\Delta\sigma_k} \left(\dot{S}^{(u)k+1}_{i+\frac{1}{2},p-\frac{1}{2}} - \dot{S}^{(u)k-1}_{i+\frac{1}{2},p-\frac{1}{2}} \right) = 0 \ . \qquad \text{(VII.4)}$$

$\mathfrak{F}^*$ is not yet defined, but everything else is known. Therefore (VII.4) determines $\mathfrak{F}^*$, except for a constant part. For the constant part, we choose

$$\overline{\mathfrak{F}^{*k}_{i,p-\frac{1}{2}}}^{\,i} = \tfrac{1}{2}\overline{F^{k}_{i,p-1}}^{\,i} \quad . \qquad \text{(VII.5)}$$

As for the coriolis force, we let $C_p = 0$.

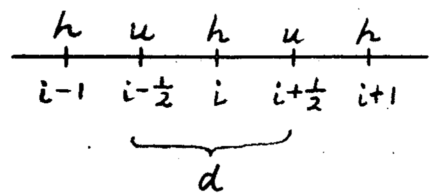
2. Averaging the pressure gradient force near the poles.

For the purposes of illustration, we consider a simple system of equations which governs a one-dimensional shallow water wave:

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0 \, , \tag{VII.6}$$

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0 \, , \tag{VII.7}$$

where the symbols are defined in section 1, chapter V. Introduce the grid, shown in the figure, which is a one-dimensional version of schemes B and C (section 1, chapter V).



Schemes B and C reduce to

$$\frac{\partial u_{i+\frac{1}{2}}}{\partial t} + \frac{g}{d}(h_{i+1} - h_i) = 0 \, , \tag{VII.8}$$

$$\frac{\partial h_i}{\partial t} + \frac{H}{d}(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) = 0 \, . \tag{VII.9}$$

Assume that

$$u_{i+\frac{1}{2}} = \hat{u}\, e^{\bar{i}k(i+\frac{1}{2})d} \, , \tag{VII.10}$$

$$h_i = \hat{h}\, e^{\bar{i}kid} \, , \quad \text{where } \bar{i} = \sqrt{-1} \, . \tag{VII.11}$$

Substituting (VII.10) and (VII.11) into (VII.8) and (VII.9), we obtain

$$\frac{d\hat{u}}{dt} + \bar{i}k\left(\frac{\sin\frac{kd}{2}}{\frac{kd}{2}}\right) g\hat{h} = 0 \, , \tag{VII.12}$$

$$\frac{d\hat{h}}{dt} + \bar{i}k\left(\frac{\sin\frac{kd}{2}}{\frac{kd}{2}}\right) H\hat{u} = 0 \, . \tag{VII.13}$$

Eliminating $\hat{h}$, we obtain

$$\frac{d^2\hat{u}}{dt^2} = k^2 c^2 \left(\frac{\sin\frac{kd}{2}}{\frac{kd}{2}}\right)^2 \hat{u} \ , \tag{VII.14}$$

where

$$c^2 = gH \ . \tag{VII.15}$$

(VII.14) is an oscillation equation, and the frequency Ω is given by

$$\Omega^2 = k^2 c^2 \left(\frac{\sin\frac{kd}{2}}{\frac{kd}{2}}\right)^2 \ . \tag{VII.16}$$

For most conditionally stable time difference schemes, the stability criterion is given by

$$|\Omega|\,\Delta t < \epsilon \ , \tag{VII.17}$$

or

$$\frac{|c|\,\Delta t}{d} \sin\frac{kd}{2} < \epsilon/2 \ , \tag{VII.18}$$

where ϵ is a constant. For the leapfrog scheme $\epsilon = 1$. $\sin\frac{kd}{2}$ has its maximum value for $kd = \pi$, or the wave length $L = 2d$. To make the scheme stable for all waves, we require that

$$\frac{|c|\,\Delta t}{d} < \epsilon/2 \ . \tag{VII.19}$$

Therefore, the smaller the grid size, the smaller must be the value of Δt.

Because the meridians converge to the poles, the grid-size in the ξ-direction becomes very much smaller than the average grid size over the globe as a whole. As a result, an extremely small Δt must be used to assure stability.

We might be able to use different Δt for the different latitudes, but that procedure would be very complicated. The usual approach is to decrease the number of grid points on the latitude circle as the latitude approaches the pole. However, it will be extremely difficult to design a space difference scheme, for such a grid, which will have the integral constraint given by (V.10). Another procedure is to keep the regular spherical grid, but use a larger space interval to compute the finite difference quotient; but this decouples each grid point from its neighboring points and introduces spurious computational modes.

The method used in our model is to smooth the ξ-component of the pressure gradient force and the divergence in the ξ-direction. Let the smoothing operator modify the amplitude of the pressure gradient and the divergence by the factor S(k). (VII.12) and (VII.13) are replaced by

$$\frac{d\hat{u}}{dt} + \bar{i}k\left(\frac{\sin\left(\frac{kd}{2}\right)}{\frac{kd}{2}}\right) g\, S(k)\, \hat{h} = 0 \,, \tag{VII.20}$$

$$\frac{d\hat{h}}{dt} + \bar{i}k\left(\frac{\sin\left(\frac{kd}{2}\right)}{\frac{kd}{2}}\right) H\, S(k)\, \hat{u} = 0 \,. \tag{VII.21}$$

The stability criterion (VII.18) then takes the form

$$\frac{|c|\,\Delta t}{d} \sin\left(\frac{kd}{2}\right) S(k) < \frac{\epsilon}{2} \,, \tag{VII.22}$$

If we choose S(k) such that

$$\sin\left(\frac{kd}{2}\right) S(k) \le \frac{d}{d^*} \,, \tag{VII.23}$$

where d^* is a specified standard length, then

$$\frac{|c|\Delta t}{d} \sin\left(\frac{kd}{2}\right) S(k) \le \frac{|c|\Delta t}{d^*} ,$$

so that the criterion for stability becomes

$$\frac{|c|\Delta t}{d^*} < \frac{\epsilon}{2} ; \qquad \text{(VII.24)}$$

This criterion depends only on the specified standard length d^*.

In the model, we let d^* be equal to the latitudinal grid size $d_\eta \equiv \frac{\Delta\eta}{n}$, which is a constant. The longitudinal grid size is $d_\xi(j) \equiv \frac{\Delta\xi}{m_j}$. Then

$$S(j,k) = \frac{d_\xi(j)}{d_\eta} \Big/ \sin\left(\frac{k}{2} d_\xi(j)\right) , \qquad \text{(VII.25)}$$

when the right hand side of (VII.25) is less than 1. Otherwise $S(j,k) = 1$.

To do this, we expand the zonal pressure gradient and the zonal mass flux into Fourier series and reduce the amplitude of each wave component by the factor $S(j,k)$. These are the bar operations shown in chapter VI. The smoothing in the mass flux given by (VI.16) is chosen to maintain the energy conservation.

It should be noted that this smoothing operation does not smooth the fields of the variables, because it is simply a generator of multiple point difference quotients. For the example that is given above, the solution of (VII.20) and (VII.21) is a neutral oscillation.

 VIII-1

VIII. TIME DIFFERENCING

1. Shortcoming of space-centered schemes.

As was pointed out in section 2, chapter V, all space-centered schemes appear to have some deficiency when the time change is in differential form. This situation is not modified by any time differencing scheme, if the space-centered differencing is used at each time level. Every space-centered difference scheme introduces averaging in either the pressure gradient force or the coriolis force, and this is the cause of the trouble.

To illustrate this difficulty in a most dramatic way, the following experiments were done by Winninghoff and Arakawa. They integrated the two-dimensional shallow water equation, keeping the coriolis force and the advection terms but no physical dissipation terms, and using scheme B and different forms of time differencing. They prescribed a point mass source and a point mass sink which were 10 grid intervals apart. The figure on the next page shows the height of the free surface, h (when the average value of 1 km is substracted), around the sink point, for four schemes of time-differencing. All of the schemes, but one (the TASU-Matsuno scheme), show a false checkerboard pattern of the free surface height instead of the monotonic slope toward the sink.

In the case of stratified flow, heat sources and sinks will have the same effect on the temperature field as the mass sources and sinks have on the free surface height field in these experiments.

h - $\bar{h}$ (in meter)

-13.	-1.	-21.	-1.	-21.	0.	-13.
0.	-30.	-3.	-42.	1.	-30.	-1.
-22.	2.	-88.	6.	-88.	-3.	-22.
-1.	-44.	6.	-356.	6.	-44.	-1.
-22.	-3.	-88.	6.	-88.	2.	-22.
-1.	-30.	1.	-42.	-3.	-30.	0.
-13.	0.	-21.	-1.	-21.	-1.	-13.

Leapfrog

2.	0.	-12.	12.	-14.	-3.	4.
-4.	-51.	16.	-120.	15.	-52.	-2.
-3.	7.	-51.	6.	-49.	7.	2.
25.	136.	26.	-350.	26.	-136.	25.
2.	7.	-49.	7.	-51.	8.	-4.
-2.	-52.	15.	-120.	16.	-51.	-4.
4.	-2.	-14.	12.	-12.	0.	2.

Lax-Wendroff

-11.	-1.	-20.	-1.	-20.	0.	-11.
0.	-31.	-3.	-45.	2.	-31.	-1.
-12.	2.	-93.	6.	-93.	-3.	-22.
0.	-46.	6.	-349.	6.	-46.	0.
-22.	-3.	-93.	6.	-93.	2.	-22.
-1.	-31.	2.	-35.	-3.	-31.	0.
-11.	0.	-22.	-1.	-20.	-1.	-11.

Matsuno

-5.	-8.	-12.	-16.	-6.	-6.	-4.
-8.	-13.	-22.	-32.	-18.	-16.	-7.
-10.	-8.	-30.	-74.	-50.	-19.	-9.
-15.	-26.	-61.	-161.	-70.	-30.	-16.
-9.	-19.	-56.	-66.	-30.	-17.	-10.
-7.	-19.	-19.	-26.	-22.	-14.	-8.
-8.	-7.	-7.	-14.	-12.	-8.	-5.

TASU-Matsuno

VIII-2

VIII-3

2. The TASU scheme (time-alternating space-uncentered scheme.)

Consider the two-dimensional, shallow water equation. For each of the directions, x and y, we use a one-sided space difference at one time level. But to obtain an overall accuracy comparable to the centered difference, we use the one-sided space difference at the opposite side at the next time level.

The TASU (time-alternating space-uncentered scheme):

At even time levels,

$$\left.\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_{ij} &\Rightarrow \frac{1}{d}\left(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\right) \\ \left(\frac{\partial h}{\partial x}\right)_{i+\frac{1}{2},j+\frac{1}{2}} &\Rightarrow \frac{1}{d}\left(h_{i+1,j+1} - h_{i,j+1}\right) \end{aligned}\right\} \text{upper-uncentered} \qquad \text{(VIII.1)}$$

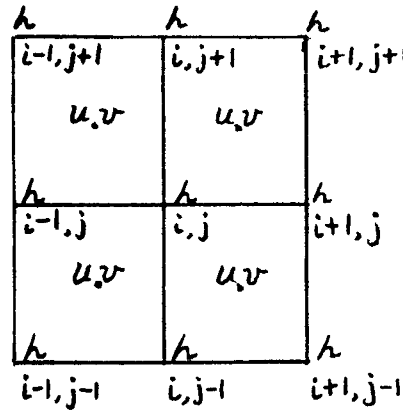
$$\left.\begin{aligned} \left(\frac{\partial v}{\partial y}\right)_{ij} &\Rightarrow \frac{1}{d}\left(v_{i+\frac{1}{2},j+\frac{1}{2}} - v_{i+\frac{1}{2},j-\frac{1}{2}}\right) \\ \left(\frac{\partial h}{\partial y}\right)_{i+\frac{1}{2},j+\frac{1}{2}} &\Rightarrow \frac{1}{d}\left(h_{i+1,j+1} - h_{i+1,j}\right) \end{aligned}\right\} \text{right-uncentered} \qquad \text{(VIII.2)}$$

At odd time levels,

$$\left.\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_{ij} &\Rightarrow \frac{1}{d}\left(u_{i+\frac{1}{2},j-\frac{1}{2}} - u_{i-\frac{1}{2},j-\frac{1}{2}}\right) \\ \left(\frac{\partial h}{\partial x}\right)_{i+\frac{1}{2},j+\frac{1}{2}} &\Rightarrow \frac{1}{d}\left(h_{i+1,j} - h_{i,j}\right) \end{aligned}\right\} \text{lower-uncentered} \qquad \text{(VIII.3)}$$

$$\left.\begin{aligned} \left(\frac{\partial v}{\partial y}\right)_{ij} &\Rightarrow \frac{1}{d}\left(v_{i-\frac{1}{2},j+\frac{1}{2}} - v_{i-\frac{1}{2},j-\frac{1}{2}}\right) \\ \left(\frac{\partial h}{\partial y}\right)_{i+\frac{1}{2},j+\frac{1}{2}} &\Rightarrow \frac{1}{d}\left(h_{i,j+1} - h_{i,j}\right) \end{aligned}\right\} \text{left-uncentered} \qquad \text{(VIII.4)}$$

 VIII-4



If there is no advection and no coriolis force, that is, if we are dealing with pure gravity waves, it can be shown that a simple use of the Euler scheme with the above time alternation is stable for small Δt. With advection and coriolis force, the simple Euler scheme becomes unstable, so that we have to combine the above alternation with some other scheme.

A scheme which combined the TASU scheme with the Matsuno scheme was tested for the example described in section 1. To explain the procedure, we write the equations symbolically in the following form:

$$\frac{dA}{dt} = f(A) \qquad \text{(VIII.5)}$$

The leapfrog scheme is

$$\frac{A^{n+1} - A^{n-1}}{2\Delta t} = f_c(A^n) \ .$$

The regular Matsuno scheme is

$$\frac{A^{(n+1)^*} - A^n}{\Delta t} = f_c(A^n) \ ,$$

$$\frac{A^{n+1} - A^n}{\Delta t} = f_c(A^{(n+1)^*}) \ .$$

VIII-5

Here f_c is a space-centered difference scheme for f.

The TASU-Matsuno scheme is

	Time	Space
$\frac{A^{(n+1)*} - A^n}{\Delta t} = f_c(A^n)$	EULER (Forward)	centered
$\frac{A^{n+1} - A^n}{\Delta t} = f_{uk}(A^{(n+1)*})$	Backward	upper-right uncentered
$\frac{A^{(n+2)*} - A^{n+1}}{\Delta t} = f_c(A^{n+1})$	EULER (Foreward)	centered
$\frac{A^{n+2} - A^{n+1}}{\Delta t} = f_{DL}(A^{(n+2)*})$	Backward	lower-left uncentered

The results obtained by these three procedures, as well as by a version of the Lax-Wendroff scheme, were shown in the figure on page VIII-2.

3. Time differencing of the model.

The upper-flux in the ξ-direction is defined by

$$(F^k_{i+\frac{1}{2},j})_{\text{upper}} = \tfrac{1}{2}\,\overline{\left(u\frac{\Delta\eta}{n}\right)}^k_{i+\frac{1}{2},j+\frac{1}{2}} (\pi_{i+1,j} + \pi_{ij}) \, . \tag{VIII.5}$$

Similarly,

$$(F^k_{i+\frac{1}{2},j})_{\text{lower}} = \tfrac{1}{2}\,\overline{\left(u\frac{\Delta\eta}{n}\right)}^k_{i+\frac{1}{2},j-\frac{1}{2}} (\pi_{i+1,j} + \pi_{ij}) \, , \tag{VIII.6}$$

$$(G^k_{i,j+\frac{1}{2}})_{\text{right}} = \tfrac{1}{2}\left(v\frac{\Delta\xi}{m}\right)^k_{i+\frac{1}{2},j+\frac{1}{2}} (\pi_{i,j+1} + \pi_{ij}) \, , \tag{VIII.7}$$

$$(G^k_{i,j+\frac{1}{2}})_{\text{left}} = \tfrac{1}{2}\left(v\frac{\Delta\xi}{m}\right)^k_{i-\frac{1}{2},j+\frac{1}{2}} (\pi_{i,j+1} + \pi_{ij}) \, . \tag{VIII.8}$$

VIII-6

Corresponding to (VI.28) and (VI.29), we also use the upper, lower, right and left pressure gradients. Also, in the right hand side of (VI.37), only one of the $(u \frac{\Delta\eta}{n})$ or $(v \frac{\Delta\xi}{m})$ terms within each bracket is picked up, with $\frac{1}{8}$ replaced $\frac{1}{4}$, when we want the uncentered expression.

We must use uncentered expressions at the same side everywhere in the system of equations at a given time level. Otherwise, an instability may occur.

The actual time-marching procedure used in the model is primarily the leapfrog scheme with a periodic use of the TASU-Matsuno scheme. At the present time (December 1971), the TASU-Matsuno scheme is used every fifth time step.

IX. LARGE-SCALE PRECIPITATION, DRY AND MOIST CONVECTION

1. Large-scale precipitation.

Large-scale precipitation occurs when $q^k_{ij} - q^{*k}_{ij}$ is the excess mixing ratio; but not all of this excess condenses, because the temperature, and therefore q^{*k}_{ij}, increases as a result of the condensation. A first guess of the condensation is taken as,

$$C\Delta t = \frac{q^k_{ij} - q^{*k}_{ij}}{1 + \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_{p_k}} \cdot \tag{IX.1}$$

$C\Delta t$ is subtracted from q^k_{ij}, and $LC\Delta t/c_p$ is added to T^k_{ij}. This process may be iterated for a better accuracy at the given step.

2. Dry convective adjustment.

When $\theta^k_{ij} < \theta^{k+2}_{ij}$ for any odd k ($\le K-2$), we assume that subgrid-scale dry convection occurs. We modify T^k_{ij} and T^{k+2}_{ij} in the following way:

$$(\Delta T^k_{ij})\,\Delta\sigma_k = -(\Delta T^{k+2}_{ij})\,\Delta\sigma_{k+2} \tag{IX.2}$$

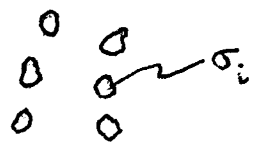
$$\theta^k_{ij} + \frac{\Delta T^k_{ij}}{\left(p^k_{ij}\right)^{\varkappa}} = \theta^{k+2}_{ij} + \frac{\Delta T^{k+2}_{ij}}{\left(p^{k+2}_{ij}\right)^{\varkappa}} \tag{IX.3}$$

This temperature change at levels k and k + 2 may cause a new unstable lapse rate at a neighboring interval. If it does, then the above adjustment is applied to that unstable interval. This process is repeated until all of the intervals are stable. In the three level model two steps are sufficient.

C-2

3. <u>Parameterization of cumulus convection</u>.

Consider an ensemble of cumulus clouds. Here, the cloud is defined as the saturated portion of the air. Let σ_i be the fractional area covered by the ith cloud in a horizontal cross-section at level z. The total fractional area



covered by all clouds is $\sigma = \sum_i \sigma_i$, where $\sum_i$ is the summation over all clouds in the unit area A.

As a basic continuity equation, we let

$$\nabla \cdot (\rho \mathbb{V}) + \frac{\partial}{\partial z}(\rho w) = 0 \ , \qquad \text{(IX.4)}$$

where ρ is a function of z only.

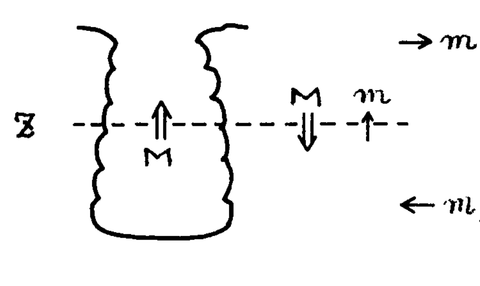
The total vertical mass flux in the cloud ensemble is given by

$$M(z) = \sum_i \int_{\sigma_i} \rho w \, d\sigma \ . \qquad \text{(IX.5)}$$

We define the large-scale vertical mass flux by

$$m(z) = \int_A \rho w \, d\sigma \equiv \overline{\rho w} \ . \qquad \text{(IX.6)}$$

$M - m$ is the net downward mass flux in the environment. The schematic cloud in the accompanying figure represents the ensemble of clouds.



Define h by

$$h \equiv c_p T + gz + Lq \ , \tag{IX.7}$$

where c_pT, gz and Lq are, respectively, enthalpy, geopotential energy, and latent heat. h is approximately conserved with respect to an air parcel, or

$$\frac{dh}{dt} = 0 \ . \tag{IX.8}$$

Combining (IX.8) with (IX.4), we have

$$\frac{\partial}{\partial t}(\rho h) + \nabla \cdot (\rho \mathbb{V} h) + \frac{\partial}{\partial z}(\rho w h) = 0 \ . \tag{IX.9}$$

We must consider entrainment and detrainment layers separately. Inside the clouds, in an entrainment layer, and immediately outside of the clouds, in a detrainment layer, we expect strong turbulent mixing. In the entrainment layer, we integrate (IX.9) over an area slightly larger than the area σ_1, at the boundary of which turbulent mixing may be ignored. Then we obtain

$$\frac{\partial}{\partial t}(\rho h_1 \sigma_1) - \left(\frac{\partial M_1}{\partial z} + \rho \frac{\partial \sigma_1}{\partial t}\right) h_e + \frac{\partial}{\partial z}(M_1 h_1) = 0 \ . \tag{IX.10}$$

where

$$h_1 \equiv \frac{1}{\sigma_1} \int_{\sigma_1} h \, d\sigma \ , \tag{IX.11}$$

$$M_1 = \int_{\sigma_1} \rho w_1 \, d\sigma \ . \tag{IX.12}$$

Here, the vertical transport of h by the internal structure of the cloud is neglected. $\partial M_1/\partial z + \rho \partial \sigma_i / \partial t$ is the entrainment rate of environment air into the cloud, which may not be steady.

In a detrainment layer, we integrate (IX.9) over an area slightly less than the area σ_1 at the boundary of which turbulent mixing may be ignored.

Then we obtain

$$\frac{\partial}{\partial t}(\rho\, h_1\sigma_1) - \left(\frac{\partial M_1}{\partial z} + \rho\,\frac{\partial \sigma_1}{\partial t}\right)h_1 + \frac{\partial}{\partial z}(M_1 h_1) = 0\ . \qquad \text{(IX.13)}$$

If we assume that the individual clouds are alike*, then the summation of (IX.10) over all clouds gives

$$\frac{\partial}{\partial t}(\rho\, h_c\sigma) - \left(\frac{\partial M}{\partial z} + \rho\,\frac{\partial \sigma}{\partial t}\right)h_e + \frac{\partial}{\partial z}(Mh_c) = 0\ , \qquad \text{(IX.14)}$$

where $h_c = h_1$. Similarly, from (IX.13),

$$\frac{\partial}{\partial t}(\rho h_c\sigma) - \left(\frac{\partial M}{\partial z} + \rho\,\frac{\partial \sigma}{\partial t}\right)h_c + \frac{\partial}{\partial z}(Mh_c) = 0\ . \qquad \text{(IX.15)}$$

Integration of (IX.9) over the environment gives

$$\frac{\partial}{\partial t}(\rho\, h_e(1-\sigma)) + \left(\frac{\partial M}{\partial z} + \rho\,\frac{\partial \sigma}{\partial t}\right)h_e - \frac{\partial}{\partial z}(M\, h_e) = -\frac{\partial}{\partial z}(\rho\bar{w}h_e) - \nabla\cdot(\rho\,\bar{\mathbb{V}}h_e)\ , \qquad \text{(IX.16)}$$

for the entrainment layer. Here $\bar{\mathbb{V}}$ is the large-scale horizontal velocity. For the detrainment layer,

$$\frac{\partial}{\partial t}(\rho\, h_e(1-\sigma)) + \left(\frac{\partial M}{\partial z} + \rho\,\frac{\partial \sigma}{\partial t}\right)h_c - \frac{\partial}{\partial z}(M\, h_e) = -\frac{\partial}{\partial z}(\rho\bar{w}h_e) - \nabla\cdot(\rho\,\bar{\mathbb{V}}\,h_e)\ . \qquad \text{(IX.17)}$$

We assume that σ is much smaller than 1. Then we obtain the following approximate equations;

for the entrainment layer:

from (IX.14), $$-\frac{\partial M}{\partial z}h_e + \frac{\partial}{\partial z}(Mh_c) = 0\ , \qquad \text{(IX.18)}$$

from (IX.16), $$\frac{\partial}{\partial t}(\rho h_e) - M\,\frac{\partial h_e}{\partial z} = -\frac{\partial}{\partial z}(\rho\bar{w}h_e) - \nabla\cdot(\rho\bar{\mathbb{V}}h_e)\ , \qquad \text{(IX.19)}$$

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*See the last paragraph of this section.

for the detrainment layer:

from (IX.15), $$M \frac{\partial h_c}{\partial z} = 0 \ , \tag{IX.20}$$

from (IX.17), $$\frac{\partial}{\partial t}(\rho h_e) + (h_c - h_e)\frac{\partial M}{\partial z} - M\frac{\partial h_e}{\partial z} = -\frac{\partial}{\partial z}(\rho \bar{w} h_e) - \nabla\cdot(\rho \mathbb{V} h_e) \ . \tag{IX.21}$$

From the definitions, $\bar{h} = h_c\,\sigma + h_e(1-\sigma)$. Because we are assuming that $\sigma \ll 1$ (and $h_c \sim h_e$), we approximate h_e by $\bar{h}$. Then,

for the entrainment layer:

$$-\frac{\partial M}{\partial z}\bar{h} + \frac{\partial}{\partial z}(M h_c) = 0 \ , \tag{IX.22}$$

for the detrainment layer:

$$M\frac{\partial h_c}{\partial z} = 0 \ . \tag{IX.23}$$

In the environment no condensation occurs.* Separating h into s and Lq, where $s \equiv c_p T + gz$, we obtain

for the entrainment layer:

$$\frac{\partial}{\partial t}(\rho\bar{s}) = M\frac{\partial \bar{s}}{\partial z} - \frac{\partial}{\partial z}(\rho\bar{w}\,\bar{s}) - \nabla\cdot(\rho\,\overline{\mathbb{V}}\,\bar{s}) \ , \tag{IX.24}$$ **

$$\frac{\partial}{\partial t}(\rho\bar{q}) = M\frac{\partial \bar{q}}{\partial z} - \frac{\partial}{\partial z}(\rho\bar{w}\,\bar{q}) - \nabla\cdot(\rho\,\overline{\mathbb{V}}\,\bar{q}) \ , \tag{IX.25}$$

for the detrainment layer:

$$\frac{\partial}{\partial t}(\rho\bar{s}) = -(s_c - \bar{s})\frac{\partial M}{\partial z} + M\frac{\partial \bar{s}}{\partial z} - \frac{\partial}{\partial z}(\rho\bar{w}\,\bar{s}) - \nabla\cdot(\rho\,\overline{\mathbb{V}}\,\bar{s}) \ , \tag{IX.26}$$ **

$$\frac{\partial}{\partial t}(\rho\bar{q}) = -(q_c - \bar{q})\frac{\partial M}{\partial z} + M\frac{\partial \bar{q}}{\partial z} - \frac{\partial}{\partial z}(\rho\bar{w}\,\bar{q}) - \nabla\cdot(\rho\,\overline{\mathbb{V}}\,\bar{q}) \ , \tag{IX.27}$$

(IX.22-23) are diagnostic equations which determine h_c as a function of height, and (IX.24-27) are prognostic equations for the large-scale temperature and

*It is also assumed that there is no evaporation of liquid water in the environment. See the last paragraph of this section.

**More exact equations are obtained by replacing s by the potential temperature, θ.

water vapor fields.

Condensation in the clouds is given by water vapor inflow from the environment, so that the environment water vapor loss, due to the flow induced by the clouds, is the amount of condensation. From (IX.25) and (IX.27), this loss is

$$[M q]_{\text{cloud base}} + \int_{\text{e.l.}} [\frac{\partial M}{\partial z} \bar{q}]\, dz + \int_{\text{d.l.}} [\frac{\partial M}{\partial z} q_c]\, dz. \qquad \text{(IX.28)}$$

Finally, we express s_c and q_c by h_c. From the definition of h,

$$h_c - \bar{h}^* = s_c + L q_c - (\bar{s} + L\bar{q}^*) .$$

Since q is saturated in the cloud and $s_c - \bar{s} = c_p(T_c - \bar{T})$, we have

$$q_c = \bar{q}^* + (\frac{\partial q^*}{\partial T})_p (T_c - \bar{T}) ,$$

and

$$s_c - \bar{s} = \frac{1}{1 + \frac{L}{c_p}(\frac{\partial q^*}{\partial T})_p} (h_c - \bar{h}^*) . \qquad \text{(IX.29)}$$

Then

$$q_c - \bar{q}^* = \frac{\frac{L}{c_p}(\frac{\partial q^*}{\partial T})_p}{1 + \frac{L}{c_p}(\frac{\partial q^*}{\partial T})_p} \frac{1}{L} (h_c - \bar{h}^*) . \qquad \text{(IX.30)}$$

For the entrainment layer, the convective warming of the environmental temperature (which is the large-scale temperature) is through $M \frac{\partial \bar{s}}{\partial z}$. This is also approximately true for the detrainment layer, because $s_c - \bar{s}$ is usually small there. The warming effect through $M \frac{\partial \bar{s}}{\partial z}$ represents the adiabatic warming due to the subsidence M in the environment, which compensates the upward mass flux M in the clouds. (See the figure at the bottom of page IX-2.) It should be realized that the heat of condensation released in the clouds is used for maintaining the excess temperature of the clouds against adiabatic cooling and entrainment of colder and drier air, and, therefore, is used for maintaining M. The heat of condensation is not directly used for warming the environmental air. (However, whether net environmental subsidence exists or not depends on the sign of $M - \mathfrak{m}$.) The factor η at a certain level in CISK models (for example, Ooyama*(1969)), may now be interpreted as the ratio of M at that level to the large-scale upward mass flux at the top of the boundary layer which is produced by the mass convergence below.

Our problem is to find M(z). If M(z) is somehow determined, we can find h_c from (IX.22) and (IX.23). Then we can find s_c (and, therefore, T_c) and q_c from (IX.29) and (IX.30). The condensation can be calculated from (IX.28). The temperature and mixing ratio of the large-scale fields are predicted by (IX.24-27).

The above discussion indicates that relating the mass flux M(z) to the large-scale fields must be, at least logically, the central core of a cumulus parameterization scheme.

It is plausible to assume that cumulus convection adjusts the environment in such a way that the energy supply from the environment to the cumulus convection is eventually terminated and M(z) becomes identically zero, unless a counteracting modification of the environment by large-scale processes exists. Here, radiation and sensible and latent heat supply from the earth's surface are included in the large-scale processes. When modification of the environment by large-scale processes exists, the environment may gain energy in the form available for cumulus convection. We then

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*Ooyama, K., 1969: Numerical simulation of the life cycle of tropical cyclones. Journ. Atm. Sci., 26, 1, pp. 3-40.

consider a quasi-equilibrium of the cumulus ensemble in which the large-scale processes, acting as forcing functions, are balanced by the convective adjustment. M(z) can be determined when we assume this balance. The feasibility of a parameterization of cumulus convection crucially depends on the existence and uniqueness of such an equilibrium state. When M(z) is found, we can estimate the characteristic cloud size in the entrainment layer provided that the relation $\frac{1}{M}\frac{\partial M}{\partial z} = \frac{2\alpha}{r}$ holds, at least approximately.

Based on the considerations outlined above, A. Arakawa (1969)* proposed a parameterization of cumulus convection for a three-level model. The current general circulation model (December, 1971) uses essentially the same parameterization.

Recently, Arakawa (1971)** presented a new parameterization, which is physically more realistic and is applicable to any number of levels. The assumption that all clouds are alike is abandoned and, instead, a spectral distribution of cumulus convection is considered. Evaporation of detrained liquid water in the environment and interactions of the cumulus convection with the subcloud layer are taken into account. We are in the process of testing this new parameterization in the general circulation model.

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* A. Arakawa, 1969: Parameterization of cumulus convection. Appendix I, Numerical Simulation of the General Circulation of the Atmosphere. Proceedings of the WMO/IUGG Symposium on Numerical Weather Prediction, Tokyo 1968, pp. IV-7 to IV-8-12.

** A. Arakawa, 1971: A parameterization of cumulus convection and its application to numerical simulation of the tropical general circulation. The Seventh Technical Conference on Hurricanes and Tropical Meteorology, December 6-9, 1971, Barbadoes.

X. SURFACE FLUXES AND PREDICTION OF GROUND CONDITIONS*

1. Surface friction.**

The surface velocity, $\mathbb{V}_s$, is estimated by a linear extrapolation of $\mathbb{V}$, with respect to σ, from levels 3 and 5 to the surface, which is level 6. The surface stress is

$$\tau_s = -\rho c_D \, |\mathbb{V}_s| \, \mathbb{V}_s \; .$$

When the surface air temperature, T_s, is equal to the ground temperature T_g, the "surface layer" is neutral, and then

over open ocean: $c_D = 0.001 \times (1 + 0.7\, |\mathbb{V}_s|)$,

with $c_D = 0.025$ as the upper limit.

over land, ice or snow: $c_D = 0.002 + 0.006 \times z_s / 5000$,

where z_s is in meters. When the surface layer is not neutral, over all surfaces,

$$c_D = (c_D)_{neut.} \, \frac{1}{1 - 7.0 \dfrac{\Delta T}{|\mathbb{V}_s|^2}}, \quad \text{for } \Delta T \equiv (T_g - T_s) < 0 \; ,$$

$$c_D = (c_D)_{neut.} \, \left(1 + \sqrt{\frac{\Delta T}{|\mathbb{V}_s|^2}}\right) \qquad \text{for } \Delta T > 0 \; ,$$

where ΔT is in $^\circ$C and $|\mathbb{V}_s|$ is in m sec^{-1}. When computing c_D, ΔT is taken as the average of the newly computed ΔT and ΔT at one time step earlier, in order to avoid oscillations in time.

2. Surface sensible heat flux.**

The surface sensible heat flux is

$$F_s = c_p \, \rho \, c_D \, |\mathbb{V}_s| \, (T_g - T_s) \; .$$

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* The formulations in this chapter were mainly done by Dr. Akira Katayama. A more detailed description will be published as a technical report, Numerical Simulation of Weather and Climate, Department of Meteorology, UCLA.

** See the footnote on the next page.

To compute T_S, we assume that F_S is equal to the flux at the top of the surface layer, which is taken as

$$- c_p \rho K((T_B - T_S) - (T_B - T_S)_{crit.}) / (z_B - z_S) .$$

We are currently using $K = 10\ m^2 sec^{-1}$. We choose $(T_B - T_S)_{crit.}$ in the following way:

$$(T_B - T_S)_{crit} = (T_B - T_S)\ \text{dry adiabatic}, \qquad \text{when } r_S = 0 ,$$

and

$$(T_B - T_S)_{crit} = (T_B - T_S)\ \text{moist adiabatic}, \qquad \text{when } r_S = 1 .$$

Otherwise $(T_B - T_S)_{crit}$ is linearly interpolated between $(T_B - T_S)_{d.a.}$ and $(T_B - T_S)_{m.a.}$ with respect to r_S, where r_S is the surface relative humidity.

3. Evaporation.

For ocean, snow and ice, the evaporation is

$$E_S = \rho c_D \mid \mathbb{V}_S \mid (q^*(T_g) - q_S) .$$

q_S is determined in a way similar to determining T_S, but without $(q_B - q_S)_{crit.}$

The evaporation from bare land is taken as

$$E_S = \beta E_{S,P}$$

where

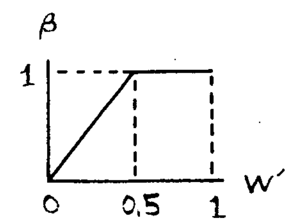
$$E_{S,P} = \rho c_D \mid \mathbb{V}_S \mid (q^*(T_g) - q_S)$$

$$= \rho c_D \mid \mathbb{V}_S \mid \left[\frac{\partial q_S^*}{\partial T} (T_g - T_S) + q_S^* - q_S \right]$$

is the potential evapotranspiration, and β is a coefficient which depends on the wetness of the ground.

** We are in the process of modification by introducing an explicit planetary boundary layer following the line proposed by J. Deardorff, Mon. Weather Rev., 100, (1972), pp. 93-106.

We let w be the amount of water in the ground, per unit mass of ground, which is available for evaporation, where w_m is its maximum possible value, and we define the ground wetness by $w' \equiv w/w_m$. For relatively large w', capillary forces are sufficient to carry the water to the surface for the potential evapotranspiration, so that we let



$\beta = 1$ when $w' \geq 0.5$,

$\beta = 2w'$ when $w' < 0.5$.

However, when $q^*(T_g) < q_S$, that is, when there is negative evaporation (dew deposit), $\beta = 1$.

4. Prediction of ground conditions.

The table on the following page shows, by a check mark, those variables which are predicted for the different types of ground condition.

Ground temperature

For sea ice, we assume a uniform thickness of 3 m, and calculate the heat conduction through the ice from the ocean below. For the prescribed permanent land ice and for the calculated snow over soil or ice, heat conduction from below is neglected. The heat capacity, C, is given by $\sqrt{\frac{\lambda c}{\omega}}$,

Predicted Variables	ocean	ice / soil or sea water	snow / ice / soil or sea water	snow / soil, & interstitial water	soil, & interstitial ice	soil, & interstitial ice & water	soil, & interstitial water
temperature, T_g		✓	✓	✓	✓		✓
wetness, w'				✓	✓	✓	✓
snow mass		✓	✓	✓	✓	✓	✓
ice mass					✓	✓	
prescribed coefficients							
latent heat, L, (cal/gm)	600	680	680	680	680	600 ~ 680	600
albedo	0.07	0.4	0.7	0.7	0.14	0.14	0.14
heat capacity, c, (cal/deg cm^2)	∞	5.1	2.3	2.3	*	*	*

*function of wetness

where λ is the thermal conductivity, c is the heat capacity per unit volume, and ω is the angular frequency of the diurnal change.

The prediction equation for T_g is

$$C \frac{\partial T_g}{\partial t} = S_S - R_S - F_S - LE_S + Q_M \ ,$$

where, at the ground surface, S_S is the net downward flux of solar radiation, R_S is the net upward flux of long wave radiation, F_S is the upward flux of sensible heat, LE_S is the upward flux of latent heat, and Q_M is the heating (or cooling) due to the freezing of water (or the melting of snow or ice). R_S, F_S and E_S depend on T_g. That part of R_S which is emitted by the ground depends on T_g and is proportional to T_g^4.

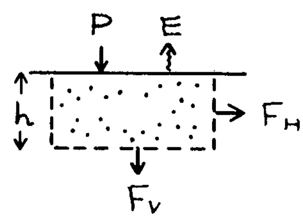
We solve the prediction equation for T_g by the backward implicit method. We do this by linearizing the equation with respect to the unknown, $\Delta T_g \equiv T_g^{n+1} - T_g^n$, where n is the index of the time level.

Snowfall, snow melting and ice melting

When $T_S < 273.1^\circ K$, we let precipitation take place as snowfall (or snow showers). The mass of snow over land or ice increases with snowfall and decreases by evaporation and melting. Melting occurs when the predicted temperature, without Q_M, is higher than $273.1^\circ K$. Then T_g is adjusted to $273.1^\circ K$ by adding negative Q_M, with a corresponding reduction in the amount of the snow mass.

Similarly, melting of the prescribed land and sea ice occurs, with the adjustment of T_g to 273.1°K by adding negative Q_M; but the mass of the ice, not being a time-dependent variable in the model, never melts completely away.

Ground wetness



For bare land,

$$\frac{\partial m}{\partial t} = P - E - F_V - F_H ,$$

where

m is the total water in the surface soil layer (g cm^{-2})

P is the rate of rainfall (g $cm^{-2} sec^{-1}$)

E is the surface evaporation (g $cm^{-2} sec^{-1}$)

F_V is the downward drainage of water through the lower boundary of the surface soil layer which holds the available water

F_H is the horizontal drainage of the available water in the soil, including the runoff at the surface

Also, $w = m/\rho h$, where ρ is the bulk density of the soil and h is the thickness of the layer which holds the available water.

From the definition of the ground wetness,

$$w' \equiv \frac{w}{w_m} = \frac{m}{w_m \rho h} \leq 1 \ .$$

Then

$$\frac{\partial w'}{\partial t} = \frac{P - E - (F_H + F_V)}{w_m \rho h}$$

The total drainage runoff, $F_H + F_V$, depends on the rate of rainfall and the wetness. The functional form currently used is

$$F_M + F_V = (P^3 + D^3)^{\frac{1}{3}} - D \ ,$$

where D is the water deficiency in the soil layer defined by

$$D \equiv (1 - w') \cdot w_m \rho h \ .$$

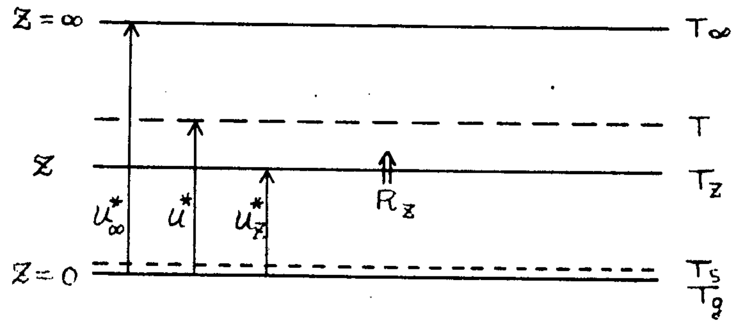
When $w' = 1$, all of the rainfall runs off. We are letting $w_m \rho h$, the total maximum available water in the ground, per unit horizontal area, be 10 gr cm^{-2}.

The ground wetness is also predicted for the soil under the snow. Both rainfall, which penetrates the snow, and snow melting contribute to an increase of the available water stored beneath the snow.

The intersticial water in the soil may freeze, and become intersticial ice, when $T_g < 273.1^\circ K$. This type of ice is carried as a prognostic variable. When T_g becomes larger than $273.1^\circ K$, it melts. This freezing and melting also affects the ground temperature.

XI. RADIATION*

1. Long-wave radiation.

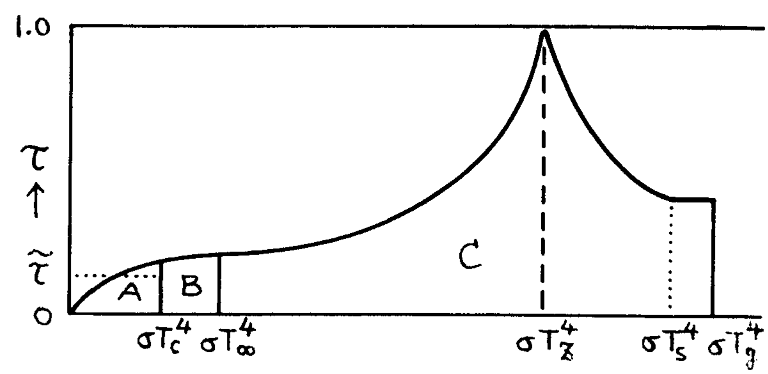


The net flux (upward positive) at the level z is given by

$$+\int_0^{T_z} \tau(u^* - u_z^*, T)\,\pi\frac{dB}{dT}\,dT + \int_{T_z}^{T_g} \tau(u_z^* - u^*, T)\,\pi\frac{dB}{dT}\,dT \quad , \qquad \text{(XI.1)}$$

where u^* is the effective amount of radiating substance integrated from the earth's surface to the reference level z, τ is the mean transmission function, averaged over the whole frequency range with the weight $d\pi B_\nu/dt$, πB_ν is the black body radiation at frequency ν, and $\pi B \equiv \sigma T^4$, the black body radiation integrated over the whole frequency range.

A typical vertical profile of τ, as a function of σT^4, is shown in the accompanying figure.



The area below the curve is the integral (XI.1).

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*The formulations in this chapter were done by Dr. Akira Katayama. A more detailed description will be published as a Technical Report, No. 6, Numerical Simulation of Weather and Climate, Department of Meteorology, UCLA.

The variation of τ is mainly through $u^* - u_z^*$ or $u_z^* - u^*$, for temperatures higher than about 220°K ($\equiv T_c$). For $T > T_c$, we can use a more or less arbitrary average temperature $\bar{T}$ for $\tau(u^* - u_z^*, T)$ or $\tau(u_z^* - u^*, T)$.

We may approximate (XI.1) by

$$R_z = A + B + C \; ,$$

where the sub-areas A, B and C are given by

$$A = \sigma T_c^4 \, \tilde{\tau}(u_\infty^* - u_z^*, T_c) \; ,$$

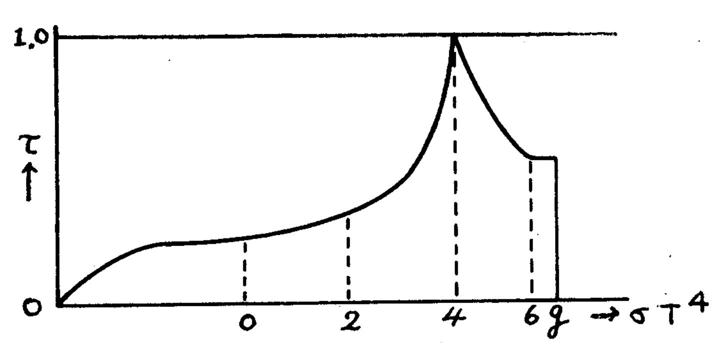
$$B = (\sigma T_\infty^4 - \sigma T_c^4) \, \tau(u_\infty^* - u_z^*, \bar{T}) \; ,$$

$$C = \int_{\sigma T_\infty^4}^{\sigma T_g^4} \tau(\,|\,u^* - u_z^*\,|\,, \bar{T}\,)\, d(\sigma T^4) \; .$$

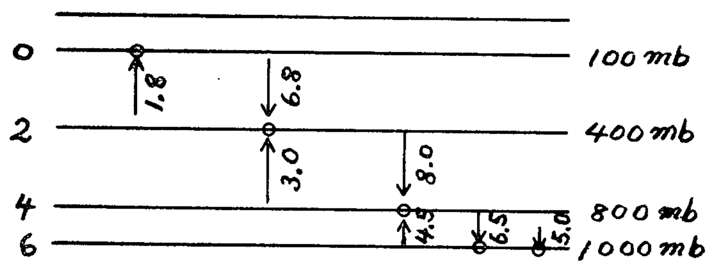
We choose $\bar{T} = 260$°K. $\tilde{\tau}$ is the mean transmission function, averaged over the whole frequency range with the weight πB_ν.

A straightforward computation of the integral C requires very high resolution near the reference level because of the sharp maximum of τ.

The computation of flux in our general circulation model is done at the even levels (0,2,4,6). Suppose we are computing the flux at level 4. For the layer 0-2, we may use the simple trapezoidal rule. But for the layers 2-4 and 4-6, we need to take special care. We assume that the average of τ in the layer 2-4, for example, can be approximated by $(1+m_{24}\tau_2)/(1+m_{24})$.



The optimum value for m was determined empirically. We found that we can safely use a constant for each layer except for small regions such as high mountains and for extreme situations such as very high or very low mixing ratio. For the vertical spacing given in the figure, the constant optimum values for m are as shown below.



A more general optimum value of m as a function of various parameters such as pressure, temperature and mixing ratio is being tested.

For the transmission functions, the following empirical formulas are currently used;

for H_2O:

$$\tau(u^*, 260^\circ K) = \frac{1}{1 + 1.75\, u^{*\,0.416}} ,$$

$$\tilde{\tau}(u^*, 220^\circ K) = \frac{1}{1 + 3.0\, u^{*\,0.408}} ,$$

for CO_2: The effective amount of CO_2 between the levels p' and p $(p' > p)$ is

$$u_{co_2}(p', p) = 127 \frac{p'^2 - p^2}{p_{oo}} , \quad p_{oo} = 1000 \text{ mb}$$

(unit: cm-NTP)

$$\tau_{co_2}(u_{co_2}) = 0.930 - 0.066 \log_{10} u_{co_2} .$$

In the 3-level model, cloudiness is either one or zero. For large-scale (non-convective) clouds, and $T < -40^\circ C$, we have the greyness factor $\alpha = 0.5$.

2. Solar radiation

Following the suggestion by Joseph* (1971), solar radiation is divided into the

$$\text{scattered part,} \quad S_o^s = 0.651 \; S_o \cos \zeta \, ,$$

and the

$$\text{absorbed part,} \quad S_o^a = 0.349 \; S_o \cos \zeta \, ,$$

where S_o is the solar constant and ζ is the zenith angle of the sun.

The absorptivity for the absorbed part is $0.271 \, (u^* \sec \zeta)^{0.303}$.

We define the function A(X) by

$$A(X) = 0.271 X^{0.303} .$$

The albedo of the atmosphere due to Rayleigh scattering is

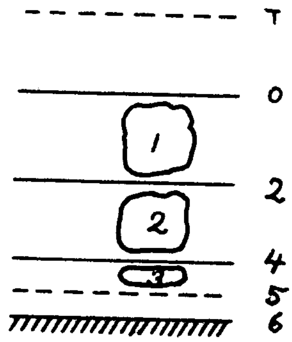
$$\alpha_o = 0.085 - 0.245 \log_{10} \left(\frac{P_s}{P_{oo}} \cos \zeta \right) ,$$

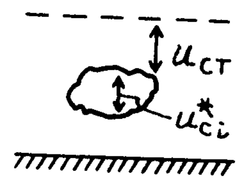
where P_s is surface pressure in mb and P_{oo} is 1000 mb.

For the albedo and absorptivity of clouds, the values used are those in the table below, as given by C. D. Rodgers (1967).

Cloud Type	Scattered part albedo (Ri')	Absorbed part albedo (Ri)	Absorbed part absorptivity (A_i)	$u^*_{c_i}$
1	0.21	0.19	0.04	0.01
2	0.54	0.46	0.20	3.0
3	0.66	0.50	0.30	12.0

* Joseph, J. H: On the calculation of solar radiation fluxes in the troposphere. Solar Energy, Vol. 13, pp. 251-261.





The equivalent water vapor amount of the cloud layer (u_{c1}^*) can be roughly determined by solving the following equation

$$A_1 \cdot S_0^a \left[1 - A(u_{CT})\right] = (1 - Ri)\, S_0^a \left[A(u_{CT} + 1.66\, u_{c1}^*) - A(u_{CT})\right]$$

Assumed values of the surface albedo (a_s) are

bare land ... 0.14 sea ... 0.07 snow ... 0.7 ice ... 0.4.

The albedo of multi-layer clouds is determined as follows:

i) 2 layers of cloud:

Let the albedos of the two clouds be R_1 and R_2. We need to consider the multiple reflection between the two cloud layers. The total transmisivity, T_{12}, is the sum of T_{12}^{I}, T_{12}^{II}, T_{12}^{III}, ...; where

$$T_{12}^{I} = (1 - R_1)(1 - R_2) ,$$

$$T_{12}^{II} = (1 - R_1)(1 - R_2)\, R_1 R_2 ,$$

$$T_{12}^{III} = (1 - R_1)(1 - R_2)\, R_1^2 R_2^2 ,$$

. . . .

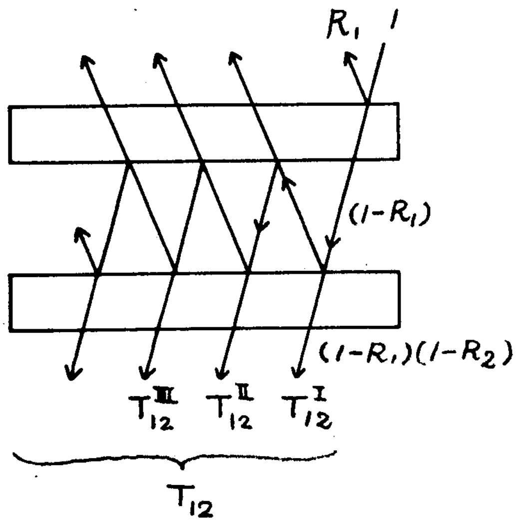
and

$$T_{12} = T_{12}^{\mathrm{I}} + T_{12}^{\mathrm{II}} + T_{12}^{\mathrm{III}} + \ldots .$$

$$= (1 - R_1)(1 - R_2)\,[1 + R_1R_2 + (R_1R_2)^2 + \ldots .]$$

$$= (1 - R_1)(1 - R_2) / (1 - R_1R_2)\ ,$$

$$R_{12} \equiv 1 - T_{12} = (R_1 + R_2 - 2R_1R_2)/(1 - R_1R_2)\ .$$



ii) 3 layers of cloud:

Let the albedos of the three clouds be R_1, R_2 and R_3. The total albedo of the 3 layers of cloud can be estimated as if they were 2 cloud layers for which the albedos are R_{12} and R_3. Then

$$R_{123} = 1 - T_{123}$$

$$= 1 - \frac{(1 - R_{12})(1 - R_3)}{1 - R_{12}R_3}$$

$$= 1 - \frac{(1 - R_1)(1 - R_2)(1 - R_3)}{1 - (R_1R_2 + R_2R_3 + R_3R_1) + 2R_1R_2R_3}\ .$$

Clear sky

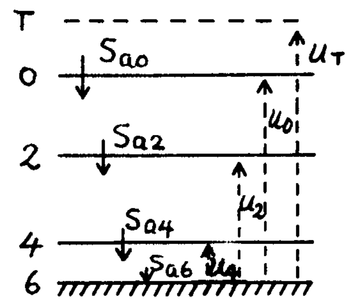
For the absorbed part of the solar radiation, the downward fluxes, S_{ai}, at the level i are

$$S_{a0} = S_0^a \, [1 - A((u_T - u_0) \sec\zeta)] \quad ,$$

$$S_{a2} = S_0^a \, [1 - A((u_T - u_2) \sec\zeta)] \quad ,$$

$$S_{a4} = S_0^a \, [1 - A((u_T - u_4) \sec\zeta)] \quad ,$$

$$S_{a6} = S_0^a \, [1 - A(u_T \sec\zeta)] \, .$$



Then, the absorption of solar radiation in the layer between i-1 and i+1, AS_i, becomes:

$$AS_1 = S_{a0} - S_{a2} \quad ,$$

$$AS_3 = S_{a2} - S_{a4} \quad ,$$

$$AS_5 = S_{a4} - S_{a6} \quad .$$

The absorption, by the earth's surface, of solar radiation reaching the surface is

absorbed part: S_{a6}

scattered part: $S_0^s \, (1 - \alpha_0)/(1 - \alpha_0 \, \alpha_s)$,

where the denominator is the correction factor due to the multiple-scattering between the atmosphere and the earth's surface. The total solar radiation absorbed by the earth's surface is

$$S_6 = (1 - a_s)\left[S_{a6} + S_0^{s} \frac{(1 - a_0)}{(1 - a_0 a_s)} \right] .$$

<u>Cloudy sky</u>

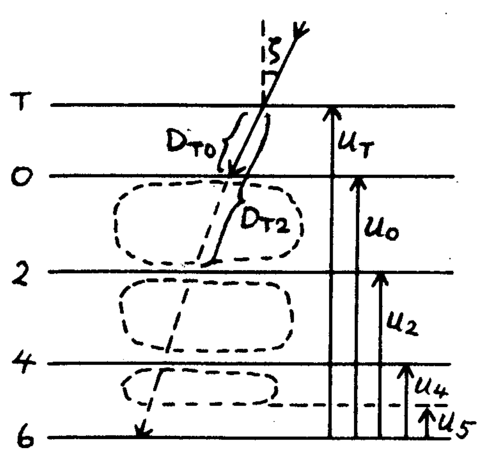
The general form for the optical length of the layer between the top of the atmosphere and level i, with the cloud layers, D_{Ti}, is

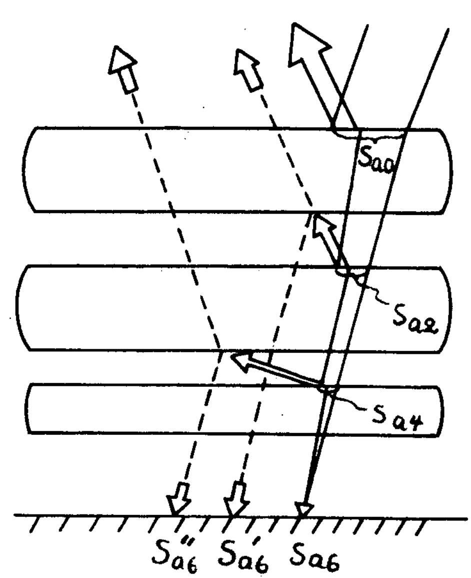
$$D_{T0} = (u_T - u_0) \sec\zeta ,$$

$$D_{T2} = D_{T0} + (1 - CL_1)(u_0 - u_2) \sec\zeta + 1.66 \cdot CL_1 \cdot u_{c1}^{*} ,$$

$$D_{T4} = D_{T2} + (1 - CL_1)(1 - CL_2) \cdot (u_2 - u_4) \sec\zeta$$
$$+ 1.66 \left[CL_2 \cdot u_{c2}^{*} + CL_1 \cdot (1 - CL_2) \cdot (u_2 - u_4) \right] ,$$

$$D_{T6} = D_{T4} + (1 - CL_3) \cdot u_4 \{ (1 - CL_1)(1 - CL_2) \sec\zeta$$
$$+ \left[(1 - CL_1) \cdot CL_2 + (1 - CL_2) \cdot CL_1 \right] \cdot 1.66 \}$$
$$+ CL_3 (u_5 + u_{c3}^{*}) \cdot 1.66 .$$





The downward solar radiation in the absorbed part, at level i ($i = 0,2,4,6$), can be expressed as

$$S_{a0} = S_0^a \, [1 - A(D_{T0})] \, ,$$

$$S_{a2} = S_0^a (1 - CR_1) \cdot [1 - A(D_{T2})] \, ,$$

$$S_{a4} = S_0^a (1 - CR_1)(1 - CR_2) \cdot [1 - A(D_{T4})] \, ,$$

$$S_{a6} = S_0^a (1 - CR_1)(1 - CR_2)(1 - CR_3)\,[1 - A(D_{T6})] \, ,$$

where $CR_i = CL_i \cdot R_i$, and $i = 1,2$ and 3. Then, the absorptions in the atmospheric layers with clouds are

$$AS_1 = (1 - CR_1)\, S_{a0} - S_{a2} \, ,$$

$$AS_3 = (1 - CR_2)\, S_{a2} - S_{a4} \, ,$$

$$AS_5 = (1 - CR_3)\, S_{a4} - S_{a6} \, .$$

In the absorbed part, some of the solar radiation, which is reflected at the tops of cloud 2 and cloud 3 ($R_2 S_{a2}$ and $R_3 S_{a4}$), can reach the earth's surface, in addition to S_{a6}, after multiple reflections between the three layers of cloud. If these amounts are defined by S'_{a6} and S''_{a6}, which refer to $R_2 S_{a2}$ and $R_3 S_{a4}$, respectively, they can be formulated as follows:

$$S'_{a6} = S_{a2} R_2 \frac{R_1 (1 - R_{23})}{1 - R_1 \cdot R_{23}} \, ,$$

$$S''_{a6} = S_{a4} R_3 \frac{R_{12} (1 - R_3)}{1 - R_{12} R_3} \, .$$

The above expressions are applied when all three layers have complete cloud coverage. To obtain more general formulae, the cloudiness must be taken into consideration. That can easily be done by using CR_i and CR_{ij} in place of R_i and R_{ij}, respectively. Then, the solar radiation in the absorbed part which reaches the earth's surface becomes

$$S_{as} = \left[S_{a3} + S_{a2} \frac{CR_2\, CR_1\, (1 - CR_{23})}{1 - CR_1 \cdot CR_{23}} + S_{a4} \frac{CR_3 CR_{12}\, (1 - CR_3)}{1 - CR_{12} \cdot CR_3} \right] \frac{1}{1 - \alpha_s \cdot CR_{123}}$$

$$CR_{ij} = (CR_i + CR_j - 2CR_i \cdot CR_j) / (1 - CR_i \cdot CR_j) \ .$$

In the scattered part, the total albedo of the clouds can be expressed by replacing R_i by R_i'. As a more general formula, we use CR_i' instead of R_i'.

$$CR_i' = CL_i \cdot R_i' \qquad (i = 1,2,3) \ ,$$

where R_i' is the albedo of cloud i for the scattered part. Then

$$CR_{123}' = 1 - \frac{(1 - CR_1')(1 - CR_2')(1 - CR_3')}{1 - (CR_1' \cdot CR_2' + CR_2' \cdot CR_3' + CR_3' \cdot CR_1') + 2CR_1' \cdot CR_2' \cdot CR_3'}$$

$$= \frac{M}{2 - M - (CR_1' + CR_2' + CR_3') + CR_1' \cdot CR_2' \cdot CR_3'} \ ,$$

where

$$M = (1 - CR_1')(1 - CR_2')(1 - CR_3') \ .$$

We assume that the total albedo of the atmosphere with clouds for the scattered part is

$$\alpha_c = 1 - (1 - CR_{123}')(1 - \alpha_0) \ .$$

The scattered part which reaches the earth's surface is

$$S_{ss} = \frac{1 - \alpha_c}{1 - \alpha_c \cdot \alpha_s} S_o^s \quad .$$

Finally, the total solar radiation at the earth's surface becomes $S_{as} + S_{ss}$. Therefore, the absorption at the earth's surface is

$$S_6 = (1 - \alpha_s)(S_{as} + S_{ss})$$

$$= (1 - \alpha_s)\left\{ \frac{1}{1 - \alpha_s \cdot CR_{123}} \left[S_{a6} + S_{a2} \frac{CR_1 CR_2 (1 - CR_{23})}{1 - CR_1 \cdot CR_{23}} + S_{a4} \frac{CR_3 \cdot CR_{12}(1 - CR_3)}{1 - CR_3 \cdot CR_{12}} \right] + \frac{1 - \alpha_c}{1 - \alpha_c \alpha_s} S_o^s \right\} .$$